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SOME THEORETICAL ASPECTS OF NONZERO SUM DIFFERENTIAL GAMES AND APPLICATIONS TO COMBAT PROBLEMS

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The theory of nonzero sum differential games (NZSDG) is extended for a class of problems in which the nonlinear system equations have bounded control variables appearing linearly. For terminal cost functions this class of problems is shown to exhibit "bang-bang" type control laws with the possibility of singular controls. A condition is derived to test for continuity of the influence functions when controls switch from a nonsingular to a singular control on singular surfaces. Two generalized forms of the transversality

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conditions are derived for NZSDG theory extending results of Dreyfus and Isaacs.

NZSDG theory is shown to be useful in modeling combat problems in which the goals of the players are not diametrically opposed. A two player and a three player penetrator-interceptor problem are presented as a NZSDG. Numerical solutions for a totally singular problem are carried out to illustrate application and a typical solution. A two player NZSDG pursuit-evasion problem is analyzed in which the cost functions of the two players are different functions of the terminal range and angle off.

SOME THEORETICAL ASPECTS OF NONZERO SUM DIFFERENTIAL GAMES AND APPLICATIONS TO COMBAT PROBLEMS

DISSERTATION

Presented to the Faculty of the School of Engineering
of the Air Force Institute of Technology

Air University

in Partial Fulfillment of the
Requirements for the Degree of
Doctor of Philosophy

by

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June 1971

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Preface

This dissertation is the result of my effort to apply nonzero sum differential game theory to practical and relevant Air Force problems. My thanks are extended to the faculty of the Air Force Institute of Technology and especially to Professor Gerald M. Anderson who has given me inspiration, guidance, and timely advice. I am indebted to the Air Force Flight Dynamics Laboratory for giving me the opportunity and time to conduct the research in this dissertation and to William L. Othling Jr. for many helpful discussions.

I am especially grateful to my wife Susan for her encouragement and patience throughout this course of study.

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Symbols

a	Superscript denoting "attacker" or a constant between 0 and 1.
$A^j(x)$	Vector function of x
A^i	Same as $A^i(x)$
$A^j{}_x$	Partial derivative of $A^j(x)$ w.r.t. x
b	Constant between 0 and 1
$B(x)$	Vector function of the state vector x
c	Superscript denoting "cooperating defender"
C^k	A scalar
C_D	Drag coefficient
C_{D_o}	Zero lift drag coefficient
C_L	Lift coefficient
$C_{L_{max}}$	Maximum lift coefficient
d	Superscript denoting "defender"
D	Drag force magnitude
e	Superscript denoting "evader"
f	n-dimensional vector function
f_x	Partial derivative of f w.r.t. x
f_U	Partial derivative of f w.r.t. U
$g(x)$	n dimensional vector function
g_x	Derivative of g(x) w.r.t. x
g	Gravity constant
h	n+1 dimensional vector function

List of Symbols and Abbreviations

h^i	Observation vector function for player i
H^i	Hamiltonian function of the i^{th} player
H^{i+}	Hamiltonian function for the i^{th} player evaluated on one side of a switching surface
H^{i-}	Hamiltonian function for the i^{th} player evaluated on the other side of a switching surface
$H^i_{U^i_j}$	Partial derivative of H^i w.r.t. U^i_j
$\dot{H}^i_{U^i_j}$	Total derivative of $H^i_{U^i_j}$ w.r.t. t
$H^i_{U^i_j U^i_j}$	Second partial derivative of H^i w.r.t. U^i_j
$\left\vert H^i_{U^i_j U^i_j} \right\vert$	Determinant of the matrix $H^i_{U^i_j U^i_j}$
J^i	Cost Function of the i^{th} player
$\bar{J}^i$	Equivalent cost function
δJ^i	Total variation in J^i
$K(U^i)$	Inequality constraint function
ℓ	Arbitrary parameter
L	Lift force magnitude
L^i	Integrand of the i^{th} player's integral cost function
m	Mass
$\mathcal{M}$	Denotes a singular surface
n	Dimension of x or load factor control variable
$n^{p(e)}$	Load factor control variable for pursuer (evader)
n_{max}	Maximum load factor
N	Number of players in a game

List of Symbols and Abbreviations

p	Superscript denoting "pursuer"
$P^i_E(t)$	Energy weighting matrix function for the i^{th} player
$P^i_F(t)$	Fuel weighting matrix function for the i^{th} player
Q^i	Weighting Matrix for the i^{th} player
R	Range
R_f	Final range
s	n dimensional parameter vector, or superscript meaning player s
s_i	i^{th} component of s
S	Reference area
sgn(y)	Signum function of y
t	Independent time variable
t_o	Initial value of the independent variable time
t_f	Final time
dt	Differential of the independent variable time
Δt_f	Induced variation of t_f
T	Thrust or final time
T^e	Thrust of evader
T^p	Thrust of pursuer
u	A scalar control variable
U^i	Control vector function of the i^{th} player
U^{i*}	Equilibrium strategy function of the i^{th} player
U^i_j	j^{th} component of the control vector function U^i
U	Set of N control vectors U^i
U*	Set of N equilibrium control vectors

List of Symbols and Abbreviations

U^j_x	Derivative of U^j w.r.t. the state vector x
$(U^*;U^i)$	Set of N equilibrium control vectors except the i^{th} which is not an equilibrium control vector
$\partial/\partial U^i_j$	Partial derivative operator w.r.t. U^i_j
v	A scalar control variable
V^i	Speed of the i^{th} player
w	A scalar control variable
W^i	Value function of the i^{th} player (a scalar)
W^i_t	Partial derivative of W^i w.r.t. t
W^i_x	Partial derivative of W^i w.r.t. the state vector x
W^i_{xx}	Second partial derivative of W^i w.r.t. x
W^i_{xt}	Second partial derivative of W^i w.r.t. x and t
$\{W^i_x\}$	The set of all partial derivatives W^i_x $(i = 1, \ldots, N)$
x	n-dimensional state vector
x_i	i^{th} component of x
x^*	state vector on an equilibrium trajectory
x_o	State vector initial condition
x_f	State vector final condition
$\bar{x}$	Augmented state vector
(x_T,y_T)	Planar coordinates of the target
(x^i,y^i)	Planar coordinates of player i
δx_f	Variation of x_f
y^i	Observation vector of the i^{th} player, or y coordinate of player i
γ	Flight path angle
γ^s	Flight path angle of player s

List of Symbols and Abbreviations

θ	Angle-off angle
θ_f	Final angle-off angle
λ^i	n-dimensional influence function vector of the i^{th} player
$\{\lambda^j\}$	Set of all λ^j vectors $j = 1, \ldots, N$
$\lambda^i_{x^d}$	The scalar component of λ^i normally associated with the state component x^d
ρ	density
σ	n-dimensional parameter vector
T	Function from $R^n \to R^1$
$\partial T/\partial\sigma$	Partial derivative of T w.r.t. σ
τ_j	Terminal surface
τ	Set of terminal surfaces
ϕ^i	Terminal cost function
$\dot{\phi}^i$	Total derivative of ϕ^i w.r.t. t
ϕ_{x_f}	Partial derivative of ϕ w.r.t. x_f
χ	Function from $R^n \to R^n$
$\partial\chi/\partial\sigma$	Partial derivative of χ w.r.t. σ
ψ	Scalar termination criteria function
ψ^c	Scalar termination function for player c
ψ^d	Scalar termination function for player d
$\dot{\psi}$	Total derivative of ψ w.r.t. t
ψ_x	Derivative of ψ w.r.t. the state vector x
ψ^{-1}	Inverse of ψ
$\dot{\psi}^{-1}$	Total time derivative of ψ^{-1}
$\delta\phi$	Total variation in ϕ

List of Symbols and Abbreviations

*	Indicates evaluation on an equilibrium trajectory
$\left\| z^{2} \right\|$	Quadratic form $z^T Q z$
$\left\| z \right\|_Q$	Square root of $z^T Q z$

Abbreviations

HJB	Hamilton - Jacobi - Bellman
NZSDG	Nonzero sum differential game(s)
TPBVP	Two point boundary value problem
w.r.t.	With respect to
ℓ.h.s.	Left hand side
r.h.s.	Right hand side

Abstract

The theory of nonzero sum differential games (NZSDG) is extended for a class of problems in which the nonlinear system equations have bounded control variables appearing linearly. For terminal cost functions this class of problems is shown to exhibit "bang-bang" type control laws with the possibility of singular controls. A condition is derived to test for continuity of the influence functions when controls switch from a nonsingular to a singular control on singular surfaces. Two generalized forms of the transversality conditions are derived for NZSDG theory extending results of Dreyfus and Isaacs.

NZSDG theory is shown to be useful in modeling combat problems in which the goals of the players are not diametrically opposed. A two player and a three player penetrator-interceptor problem are presented as a NZSDG. Numerical solutions for a totally singular problem are carried out to illustrate application and a typical solution. A two player NZSDG pursuit-evasion problem is analyzed in which the cost functions of the two players are different functions of the terminal range and angle off.

I. Introduction

Isaacs [17] introduced differential game theory in 1954, and since that time many researchers have investigated the theory for application to combat problems between two vehicles. (A fairly complete reference list is compiled in ref. [15]). The theory introduced by Isaacs is a zero sum theory- so called because the goals of the two players are assumed to be precisely opposite. Many conflict situations, most notably pursuit-evasion type conflicts, are adequately modeled by this theory, the result being simultaneous "optimal" solutions for each player. However, there are aspects of the pursuit-evasion differential game that cannot be modeled by the zero sum theory. When the goals of the two players are not diametrically opposed or if there are more than two vehicles in the conflict, each with a different goal, then a more general theory is required. Recent investigations [8, 27, 33] of nonzero sum differential game (NZSDG) theory have shown this theory to be much more general, and in fact, zero sum differential game and optimal control theory can be considered as subclasses of NZSDG theory. This NZSDG theory can be used to model two player combat problems in which the goals of the players are not precisely opposite, and problems in which there are more than two players.

One of the primary obstacles in application of NZSDG theory to practical problems is the fact that the influence functions in the

generalized Euler-Lagrange equations and the partial derivatives of the value function in the generalized Hamilton-Jacobi-Bellman partial differential equations can be discontinous when controls are discontinuous. This problem does not occur in optimal control problems and rarely occurs in zero sum differential game problems [9]. If these discontinuities do occur in a problem, the solution becomes quite difficult since special conditions such as the Weierstrass-Erdmann corner conditions must be employed to calculate the discontinuities.

Purpose of Dissertation

The purpose of this dissertation is to extend NZSDG theory to enable solutions for a class of combat problems involving two or more vehicles.

The original work in this dissertation consists of: (1) generalizing to NZSDG theory the transversality conditions of Isaacs' [17] zero sum differential game theory and of Dreyfus' [10] optimal control theory. This generalization is desired since these forms of the transversality conditions are familiar and easy to apply. (2) developing a theorem which tests whether the influence functions are continuous at the junction of nonsingular trajectories with singular surfaces for a class of terminal cost problems with nonlinear state equations and bounded linear controls. This theorem allows one to test a problem in the class to determine if the influence functions will be continuous on singular surfaces. Influence function continuity considerably eases the obtaining of solutions. (3) applying the theorem above and NZSDG theory to NZSDG penetrator-interceptor problems and to a two player

NZSDG pursuit-evasion problem.

Guide to Subsequent Chapters

In Chapter II the mathematical background and necessary theory is presented. Chapter III defines the class of problems considered in this dissertation and the influence function continuity theorem is stated and proved. Chapter IV presents a two player and three player NZSDG interceptor-penetrator problem. The solutions and control laws are characterized, and for the totally singular problem numerical solutions are obtained. Chapter V defines a two-aircraft NZSDG pursuit-evasion problem in which the goals of the players are not precisely opposite. The problem is posed as a fixed terminal time, terminal cost problem. The control laws and singular surfaces are characterized for the general problem, and two special cases are examined by finding some backward solutions.

The significant contribution of this dissertation is to show that NZSDG theory can be employed to model combat problems with two or more combatants and that equilibrium solutions can be obtained from the theory's application; also that the use of NZSDG theory results in a more general problem and provides more flexibility and realism in modeling the goals of each combatant.

II. Nonzero Sum Differential Game Theory

This chapter presents the definitions, concepts, and theory of a deterministic NZSDG. Necessary conditions for an equilibrium solution and special conditions on switching surfaces are presented. The results of Case, Ho, Prasad, Ragade, Sarma, and Starr Ref. [18, 14, 15, 16 27, 28, 31, 33] are the primary sources for the material in this chapter.

Problem Formulation

Essential to every differential game problem are three entities

i. Players

ii. Cost Functions

iii. Information Sets

The role of each of these entities in NZSDG theory is discussed below. The state of the N Players in a NZSDG is governed by the vector differential equation

$$\dot{x} = f\ (x,\ t,\ U) \qquad x\ (t_o) = x_o \tag{2.1}$$

where x is the n-dimensional state vector containing the state components of the N players, t represents time, and

$$U = (U^1,\ U^2,\ .\ .\ .\ .,\ U^i,\ .\ .\ .\ .,\ U^N) \tag{2.2}$$

where U^i is the control vector of the i^{th} player, and generally player i's choice of U^i is constrained to a constraint set Ω^i. The state x evolves from the initial state x_o to some final state $x(t_f)$ where $x(t_f)$ lies on an n-dimensional terminal manifold

$$\psi\ [x(t_f),\ t_f] = 0 \tag{2.3}$$

An n-dimensional terminal surface in the n + 1 dimensional space of x and t is chosen because of its analytic tractability. The final time t_f is free unless otherwise specified. The manner in which x evolves is determined by Eq (2.1) and the players' choice of controls.

The cost function of the i^{th} player is

$$J^i = \phi^i [x(t_f), t_f] + \int_{t_o}^{t_f} L^i [x(t), t, U] \, dt \qquad i = 1, \ldots, N \tag{2.4}$$

See Appendix A for a discussion of meaningful cost functions.

Finally, each player makes his control decisions based upon the information available to him. For our purpose information can be placed in two categories:

(1) knowledge of the capabilities and goals of all the players, and

(2) knowledge of the state x.

Perfect information in category (1) implies each player knows the state equation Eq (2.1), the termination criteria Eq (2.3), and the cost functions Eq (2.4). The extent of information in category (2) is usually represented by observation equations so that the i^{th} player's state information y^i is given by

$$y^i = h^i (x, t) \qquad i = 1, \ldots, N \tag{2.5}$$

Each player selects his control according to a rule (control law) based on the observations,

$$u^i = U^i (y^i, t) \qquad i = 1, \ldots, N \tag{2.6}$$

The function U^i is called player i's strategy. When player i's state information is perfect $y^i \equiv x$, and Eq (2.6) becomes

$$u^i = U^i (x, t) \tag{2.7}$$

Eq (2.7) is called a closed-loop control law. On the other hand, if no state information except x_o is known by player i, then $y^i \equiv x_o$ and the control law

$$u^i = U^i (x_o, t_o, t) \tag{2.8}$$

is called an open-loop control law. These two control laws will be considered in this dissertation along with another control law to be discussed later called an "open-loop feedback control law".

Solution Concepts

Three solution concepts exist in the NZSDG theory: (1) equilibrium, (2) mini-max (security), and (3) noninferior (Pareto optimal). This dissertation is concerned only with the equilibrium solution; however, the two remaining solutions will be discussed briefly.

Equilibrium Solution

In the equilibrium solution each player's goal is to minimize his own cost function, thus making this solution a noncooperative one. The following mathematical definitions and equations define the solution. Define the set of equilibrium strategies U* and the set $(U^*; U^i)$ as

$$U^* \equiv (U^{1*}, \ldots, U^{N*}) \tag{2.9}$$

and

$$(U^*, U^i) \equiv (U^{1*}, \ldots, U^{i-1*}, U^i, U^{i+1*}, \ldots, U^{N*}) \tag{2.10}$$

For the NZSDG formulation in Eqs (2.1) - (2.8), if there exists a strategy set U* such that

$$J^i (U^*) \leq J^i (U^*; U^i) \qquad i = 1, \ldots, N \tag{2.11}$$

then U* is said to be an equilibrium strategy. The trajectory

$$x^* = x (x_o, t_o, U^*) \tag{2.12}$$

is called an equilibrium trajectory.

Mini-Max (Security) Solution

One of the categories of information discussed earlier in this chapter is knowledge of all players' cost functions. If this information is not available to each player, then a player may adopt a conservative viewpoint and assume all other players are opposing him. Such a strategy is called a mini-max (security) strategy. Starr [33] points out that this solution is equivalent to solving N two player zero-sum games wherein for the i^{th} game player i selects a strategy to minimize J^i while all other players select strategies to maximize J^i. Once player i has solved his zero sum game $(i = 1, \ldots, N)$ to determine his strategy, the N strategies are employed. According to Starr [33], the resulting trajectory is generally surprising to each player because of the conservative approach taken by each. For a further discussion of this solution, consult References [27, 28, 33, 35].

Noninferior (Pareto Optimal) Solution

If a negotiated or cooperative solution can be agreed upon by all the players (such as in an economic situation), then all costs are less than or equal to the corresponding costs of the equilibrium solution. A noninferior solution has the property that any other solution which gives a better result for one player also gives a worse result for another player. Any negotiated solution should be chosen from the set of noninferior solutions [35]. Consult References [27, 28, 33, 35] for a further discussion of this solution concept.

Necessary Conditions for the Equilibrium Solution

Bellman's Dynamic Programming (or principle of optimality) and Pontryagin's minimum principle can be generalized to the NZSDG problem

to give two different necessary conditions for an equilibrium solution. Bellman's principle results in a set of generalized Hamilton-Jacobi-Bellman (HJB) partial differential equations which the equilibrium solution must satisfy. Furthermore, Bellman's principle requres the assumption of perfect information [27] so that the control laws are closed-loop laws as in Eq (2.7). Pontryagin's minimum principle on the other hand is usually associated with open-loop control laws as in Eq (2.8), and this principle leads to a set of generalized Euler-Lagrange equations (called influence function equations in this dissertation) which together with the state equation Eq (2.1) are the "Characteristic" equations for the HJB partial differential equations. Thus the HJB equations constitute a much more general necessary condition.

In optimal control and zero sum differential game problems the state information available to the players has no effect on the form of the influence function equations, but the state information assumed in a NZSDG problem can have a marked effect on the form of these equations. This effect will be made clear in the following sections. Because the HJB equations are seldom solvable except for unconstrained quadratic cost linear dynamics problems, all solutions are generally obtained from the influence function equations. The main utility of the HJB equations is to verify that an equilibrium solution candidate obtained from the influence function and state equations satisfies the HJB equations. This satisfaction of the HJB equations is another necessary condition which an equilibrium solution must meet.

Hamilton-Jacobi-Bellman Partial Differential Equations

Consider the equilibrium strategy U* of Eq (2.11). Define the value W^i for player i to be the cost of player i on an equilibrium trajectory with the general intial point (x, t),

$$W^i(x, t) \equiv \phi^i[x^*(t_f), t_f] + \int_t^{t_f} L^i[x^*(\tau), \tau, U^*]\, d\tau$$

$$i = 1, \ldots, N \qquad (2.13)$$

Restricting U* to be piecewise continuous we see that W^i is continuous and piecewise differentiable. The values can be shown to satisfy the coupled system of partial differential equations hereafter referred to as the HJB equations [27],

$$W^i_t = -\min_{U^i} H^i(x, W^i_x, t, U^*; U^i) \qquad i = 1, \ldots, N \qquad (2.14)$$

where

$$H^i \equiv L^i(x, t, U) + W^i_x f(x,t,U) \qquad (2.15)$$

with the boundary condition given on the terminal manifold

$$W^i[x(t_f), t_f] = \phi^i[x(t_f), t_f] \qquad (2.16)$$

If the minimization operation in Eq (2.14) is carried out subject to all constraints, the equilibrium controls U^{i*} are found to depend functionally on W^i_x, x and t

$$U^{i*} = U^{i*}(W^i_x, x, t) \qquad i = 1, \ldots, N \qquad (2.17)$$

Substituting Eq (2.17) into Eq (2.14) we obtain another form for the HJB equations -- a coupled nonlinear system of partial differential equations

$$W^i_t(x, t) = -H^i[x, t, \{W^j_x\}] \qquad (2.18)$$

$$i, j = 1, \ldots, N$$

with the boundary condition given by Eq (2.16).

The HJB equations are defined except on surfaces in the state space on which the controls are discontinuous. For a rigorous development of the HJB equation see Prasad [27] and Sarma [31].

The advantage of a solution to the HJB equations is clear, for if an analytic expression is available for $W^i(x, t)$ and $W_x{}^i$ then from Eq (2.17) one sees that a closed loop control law is obtained. Solving for the values W^i is equivalent to finding a field of equilibrium solutions. Unfortunately, one is generally required to settle for less if the problem dynamics are constrained or nonlinear. The solution of the HJB equations by the method of characteristics [10] is the alternative. The characteristic differential equations for Eq (2.18) turn out to be the state equation Eq (2.1) and the influence function equations from Pontryagin's minimum principle.

<u>Influence Function Differential Equations</u>

Assume that the information set of each player is perfect which implies a closed-loop control law for each player; then the necessary conditions which must be satisfied on an equilibrium trajectory are the state equation

$$\dot{x} = f\ (x, t, U^*) \qquad\qquad x\ (t_o) = x_o \tag{2.19}$$

and the influence function equations

$$\dot{\lambda}^i = -\ (H^i_x + \sum_{\substack{j = 1 \\ j \neq i}}^{N} H^i{}_{U^j}\ U^j{}_x)^* \qquad\qquad i = 1, \ldots, N \tag{2.20}$$

where

$$H^i = L^i\ (x, t, U) + \lambda^i\ f\ (x, t, U) \tag{2.21}$$

and U^{i*} is the admissible minimizing control in the equation

$$H^{i*} = \min_{U^i} H^i\,(x,\ \lambda^i,\ t,\ U^*\ ;\ U^i) \tag{2.22}$$

The summation term in Eq (2.20) is absent in optimal control and zero sum differential game problems, and as we will see in later chapters, varies according to the information set assumed. For example, if perfect observations are assumed as in Eq (2.7), ${U^j}_x$ is generally nonzero in Eq (2.20); however, if no observation other than initial conditions are assumed as in Eq (2.8) then ${U^j}_x$ is identically zero. The influence function λ^i is identical to the partial derivative of the value ${W_x}^i$, when ${W_x}^i$ is evaluated along an equilibrium trajectory. The essential difference between λ^i and ${W_x}^i$ is that λ^i is a function of time only and ${W_x}^i$ is a function of both state x and t regarded as instantaneous initial conditions. Mathematically the relationship may be written

$$\lambda^i\,(t) = {W_x}^i\,[x^*\,(t),\ t] \tag{2.22}$$

where x^* (t) evolves according to the state equation Eq (2.19).

It is important to note that the influence equations are valid over the same regions as the HJB equations and solutions of the influence function equations and state equation must satisfy the HJB equations. The influence function equations are well defined except on surfaces in the state space where the controls U^i are discontinuous. Appendix B presents a formal derivation of the influence function equations from the HJB equations.

The boundary conditions for the influence function equations Eq (2.20) are specified in terms of the state and time on the terminal manifold, and these boundary conditions are generally called transversality conditions. Several forms of the transversality conditions exist; however in this dissertation we will use a generalized form of Dreyfus's

transversality condition because of its ease of application. The generalization of the Dreyfus [10] form of the transversality conditions along with a generalization of the familiar Isaacs [17] form are presented in Appendix C. Also in Appendix C is presented the more general Berkovitz form [5] due to Sarma [31]. The Isaacs form and Berkovitz form are presented since both are in general use.

The Dreyfus form requires a terminal cost function of the form

$$J^i = \phi^i [x (t_f), t_f] \qquad i = 1, \ldots, N \tag{2.24}$$

so that $L^i = 0$ in Eq (2.4). This is not overly restrictive since every problem with a cost function containing an integral can be easily converted to an equivalent terminal cost problem [17]. See Appendix C for the details of this conversion.

The Dreyfus form of the transversality conditions is

$$\lambda^i (t_f) = [\phi^i_x - (\dot{\phi}^i/\dot{\psi}) \psi_x]_{t = t_f} \tag{2.25}$$

where $\dot{\phi}$ and $\dot{\psi}$ are total time derivatives.

Since the state equation boundary conditions are specified at t_o and the influence function boundary conditions are specified at t_f, a two point boundary value problem (TPBVP) results. Assuming that a solution exists, we see that solving the TPBVP is equivalent to finding a candidate for the equilibrium solution to the original NZSDG; hence the NZSDG problem has been transformed into an equivalent TPBVP. Solutions to the TPBVP are called candidates for an equilibrium solution since the TPBVP is a necessary but not sufficient condition.

Switching Surfaces

When we attempt to solve the TPBVP consisting of the state and influence function equations Eqs (2.19) and (2.20), we often find that the equilibrium controls are discontinuous, especially if there are control constraints present. These discontinuities in control occur on switching surfaces in the state-time space, the surfaces being classified as transition, singular, dispersal and abnormal.[1] The HJB and influence function equations are generally not defined on these switching surfaces, and the partial derivatives W^i_x, W^i_t and the influence functions λ^i can be discontinuous when the trajectory crosses or enters these switching surfaces.

Special conditions must be satisfied on these surfaces which serve to continue solutions across or along the surfaces. The following sections describe the surfaces and give conditions for their construction.

Transition Surface

The transition surface is one on which the generalized Weierstrass-Erdmann corner condition holds [28]. Let (+) and (-) denote one-sided limits on respective sides of the transition surface, then the following corner condition holds [28], provided that the trajectory is not parallel to the surface:

1. The names "transition" and "dispersal" are due to Isaacs [17] while "abnormal" surfaces are surfaces like Isaacs' "barrier". Abnormal surfaces are discussed in Ref. [27].

$$(H^{i+} - H^{i-})\, dt - (\lambda^{i+} - \lambda^{i-})\, dx = 0 \qquad i = 1, \ldots, N \tag{2.26}$$

The variations dx and dt are arbitrary variations on the transition surface. If the transition surface is parametrized, say by the n - dimensional parameter σ where

$$\sigma \equiv (\sigma_1, \sigma_2, \ldots, \sigma n)$$

such that the surface is described by the parametric equations

$$t = T(\sigma) \qquad x = \chi(\sigma)$$

then Eq (2.26) takes the form [28]

$$(H^{i+} - H^{i-})\, \partial T/\partial\sigma - (\lambda^{i+} - \lambda^{i-})\, \partial\chi/\partial\sigma = 0 \qquad i = 1, \ldots, N \tag{2.27}$$

In Reference [28] it is proven that if all controls except U^{i*} are continuous on player i's transition surface and the trajectory is not parallel to the surface, then λ^i, the i^{th} players influence function vector, is continuous on the surface (remember that the influence functions for the other players λ^j, $i \neq j$, may be discontinuous).

Singular Surface

Singular surfaces consist of a family of trajectories on which for at least one player, say the i^{th}, a control component ${U^i}_j^*$ is on the interior of its constraint set for a nonzero time interval and the coefficient of ${U^i}_j$ in the Hamiltonian function is identically zero, so that

$$H^i_{U^i_j} = 0 \qquad |H^i_{U^i\, U^i}| = 0 \tag{2.28}$$

on this time interval. The most common and important case occurs when the control U^i_j appears linearly in the state equation and the Hamiltonian. We will consider only this linear case. For this case Eq (2.28) does not suffice to determine the minimizing (singular) control since U^i_j does not appear in Eq (2.28). A higher order necessary condition called the generalized Legendre Clebshe condition [10, 29] is required to test the singular control which is generally determined by Eq (2.29) below. Although the above references developed the Legendre-Clebsh condition only for optimal control problems the result has been extended to zero sum differential games [1] and to NZSDG [28]. Since $H^i_{U^i_j}$ is identically zero on the singular arc, so are all of its time derivatives, thus

$$\dot{H}^i_{U^i_j} = \ddot{H}^i_{U^i_j} = . . . = 0 \qquad (2.29)$$

Successive differentiation as in Eq (2.29) with substitution of the state and influence function equations for player i generally results in an equation containing U^i_j explicitly. Robbins [29] shows that if U^i_j appears at all in Eq (2.29) it first appears explicitly in an even time derivative of $H^i_{U^i_j}$. Not only does Eq (2.29) often provide an equation which determines U^i_j, but it also provides useful relations among the state and influence function variables which must hold along the singular arc. The necessary condition for a singular control U^i_j to be an admissible equilibrium control is the satisfaction of the inequality

$$(-1)^k \, \partial/\partial U^i_j \, [\, d^{2k}/dt^{2k} \, H^i_{U^i_j} \,] \geq 0 \qquad (2.30)$$

Robbins [29] calls the quantity 2k the "degree" of the singularity where

$$d^{2k}/dt^{2k} \; H^i_{U^i_j} = 0 \tag{2.31}$$

is the equation in which U^i_j first appears explicitly.

It is possible that more than one control component is singular on the same time interval. In this case Eq (2.30) is applied to each control component. If one of the control components say U^i_j has a lower degree of singularity, when U^i_j first appears in Eq (2.30) the functional relationship between U^i_j and the state and influence function variables is determined. This result is substituted back into Eq (2.30) to eliminate U^i_j from the equation then successive differentiation is again employed to find the control component with the next higher degree of singularity.

Jacobson and Speyer [19, 20, 32], and McDannell and Powers [23] have derived new necessary conditions and new sufficient conditions for totally singular optimal control problems; however, it is not known whether these conditions can be readily extended to zero sum and nonzero sum differential games.

Dispersal Surface

A dispersal surface for player i is a surface from which player i can play more than one equilibrium strategy without changing his cost. Examples of this type of surface are found in Isaacs' Homocidal Chauffer Game and Game of Two Cars [17]. The following equation characterizes the dispersal surface for player i [27, 28]:

$$H^i\,(x, \lambda^i_1, U^i_1, t)\,dt - \lambda^i_1\,dx = H^i\,(x, \lambda^i_k, U^i_k, t)dt - \lambda^i_k\,dx \tag{2.32}$$

$$k = 2, \ldots, p$$

where p is the number of equivalent strategies and k indicates the k^{th} equivalent strategy. dx and dt are arbitrary variations on the dispersal surface. The dispersal surface can occur in pursuit-evasion problems. A typical situation illustrating the nature of the surface occurs when player i has the option of making an equilibrium control choice of either a turn right or a turn left. As its name "dispersal surface" implies, trajectories generally diverge from the surface.

Abnormal Surface

In this dissertation normal problems have been assumed; however, examples of abnormal surfaces appear in the literature and need to be mentioned. The main examples of such surfaces are Isaacs' semipermeable surfaces with the barrier surface a special case [17]. Prasad [27] points out that another abnormal surface is Isaacs' Equivocal Surface which is a dispersal surface for one player and a singular surface for the other. The reader interested in abnormal surfaces is referred to Ref [27].

This chapter has presented the background and theory of NZSDG to acquaint the reader with the topic. The two approaches to necessary conditions provide the HJB equations and the influence function equations. In practice, for nonlinear or constrained problems it is usually required to solve the TPBVP consisting of the influence function equations and state equation in order to obtain equilibrium solutions. The conditions necessary to construct the various switching surfaces which are required in solving the TPBVP were presented. It should be noted however, that these conditions are generally difficult to apply.

III. A Class of Nonlinear NZSDG Problems with Controls Appearing Linearly

A broad class of differential game problems involves a nonlinear dynamical system with controls appearing linearly in the state equation. This chapter treats the terminal cost NZSDG which results from the above type of dynamics with constrained controls. The following sections define the problem class, formulate the differential game, HJB equations, influence function equations, TPBVP, define the control switching functions, and present conditions under which the influence functions are continuous on singular surfaces. The singular surface occurs frequently in many problems and is often the only switching surface involved in the problem; therefore, it is important to know the behavior of the influence functions on this surface. If the influence functions are continuous at the junction of a trajectory and a singular surface the solution of the TPBVP is considerably easier.

Chapters IV and V are devoted to two examples from this class of problems.

Problem Statement

For the class of problems considered in this chapter the state equation is

$$\dot{x} = g\ (x) + \sum_{j=1}^{N} A^j\ (x)\ U^j = f\ (x,\ U) \qquad x\ (t_0) = x_0 \tag{3.1}$$

The control variable, U^i corresponding to the i^{th} player is a scalar and is constrained such that the following inequality holds

$$|U^i| \leq 1 \qquad i = 1,\ .\ .\ .,\ N \tag{3.2}$$

(The problem can be generalized for vector controls with the major effect being notational complexity). The state vector x contains the state components of the N players.

The goal of each player is to choose an admissible control strategy U^{i*} such that his own terminal cost function

$$J^i = \phi^i [x(t_f), t_f)] \tag{3.3}$$

is minimized. $x(t_f)$ and t_f satisfy the termination criteria

$$\psi [x(t_f), t_f] = 0 \tag{2.3}$$

We assume perfect information which implies closed-loop strategies for each player; however, it will be shown for this class of problems that the closed-loop and open-loop control laws will result in the same influence function equations hence the same equilibrium solutions.

Formulation of the HJB Equations, Influence Function Equations and the TPBVP

We can formulate the HJB equations,

$$W_t^i (x, t) = - \min_{U^i} H^i (x, t, W_x^i, U^*; U^i) \qquad i = 1, \ldots, N \tag{2.14}$$

$$W^i [x(t_f), t_f] = \phi^i [x(t_f), t_f] \qquad i = 1, \ldots, N \tag{2.16}$$

where H^i is given by

$$H^i = W_x^i f = W^i{}_x [g(x) + \sum_{j=1}^{N} A^j (x) U^j] \tag{3.4}$$

The admissible U^{i*} which minimizes H^i in Eq (3.4) (holding the other controls U^{j*}, $j \neq i$, at their equilibrium values) is (by inspection of Eq (3.4))

$$U^{i*} = - \operatorname{sgn} (W_x{}^i A^i) \tag{3.5}$$

where

$$\operatorname{sgn}(y) = 1 \qquad y > 0$$
$$\qquad\quad -1 \qquad y < 0 \tag{3.6}$$

Eq (3.5) holds except when the argument $W^i_x A^i$ satisfies the equation,

$$W^i_x A^i = 0 \tag{3.7}$$

over a nonzero time interval. In this event it is possible for U^{i*} to be singular, and the necessary conditions given by Eqs (2.28) through (2.31) must be applied to determine the control U^{i*} and test its admissibility as a candidate for an equilibrium control.

We can now write the HJB equations for this problem in the form

$$W^i_t = - W^i_x \left[g(x) + \sum_{j=1}^{N} A^j U^{j*}\right] \tag{3.8}$$

with the boundary condition given by Eq (2.16). Eq (3.8) is a system of first order nonlinear coupled partial differential equations where the equations are coupled through the last term of the r.h.s.. This equation cannot, in general, be solved in closed form; however, it is an additional necessary condition since the state and influence function equations are the characteristic equations for the HJB equations and therefore must satisfy it.

The influence function equations for this class are

$$\dot{\lambda}^i = - \lambda^i \left(g_x + \sum_{j=1}^{N} A^j_x U^{j*} + \sum_{j=1}^{N} A^j U^{j*}_x \right)$$
$$i = 1, \ldots, N \tag{3.9}$$

The terminal boundary conditions for Eq (3.9) are given by Eq (2.25). In Eq (3.9) U^{j*} is given by

$$U^{j*} = - \operatorname{sgn}(\lambda^j A^j) \tag{3.10}$$

or in the event the argument $\lambda^j A^j$ satisfies the equation

$$\lambda^j A^j = 0 \tag{3.11}$$

over a nonzero time interval, the control U^{j*} may be singular and

must be determined by the necessary conditions, Eqs (2.28) through (2.31). In Eq (3.9) if $j = i$, $\lambda^i A^i U_x^{i*}$ is identically zero (see Appendix B). In any event, the expression U_x^{j*} in the last term in the r.h.s. of Eq (3.9) is either identically zero in regions where the controls are constant or undefined on switching surfaces. In solving the influence function equations, the special conditions for the various surfaces outlined in Chapter II must be used to continue solutions across the switching surfaces. In the event a trajectory lies in a switching surface such as in the case of a singular control segment, special arguments to be discussed later in this chapter must be employed.

The equation for the influence functions holding between switching surfaces, can now be written

$$\dot{\lambda}^i = - \lambda^i \left(g_x + \sum_{j=1}^{N} A^j U_x^{j*}\right) \qquad i = 1, \ldots, N \tag{3.12}$$

Control Laws

Eqs (3.5) and (3.10) defining the control U^{i*} are known as a "bang-bang" control law, so that the controls for this class of problems are "bang-bang" with the possibility of singular controls when Eqs (3.7), or (3.11) holds. The development thus far in this chapter assumes perfect information, but now let this requirement be relaxed. Suppose no state observations except $x(t_o)$ are permitted. Then, according to Eq (2.8),

$$u^i = U^i (x_o, t_o, t) \tag{2.8}$$

Hence U_x^i in the influence function equations Eq (3.9) is identically zero and the influence function equations for the open-loop control law problem become

$$\dot{\lambda}^i = - \lambda^i \left(g_x + \sum_{j=1}^{N} A^j U_x^{j*}\right) \qquad i = 1, \ldots, N \tag{3.13}$$

which are identical to the closed-loop influence function equations Eq (3.12) for this class. Thus the open-loop trajectory is identical to the closed-loop trajectory for a given set of initial conditions. This identity is caused by the control constraints Eq (3.2) and the linearity of the controls in the state equation Eq (3.1) and is not dependent on the information pattern. Because the open-loop and closed-loop control laws result in identical trajectories, in this class of problems a sampled data feedback solution (i.e., an open-loop solution using each new sample point as an updated initial condition) approaches the closed-loop solution in the limit where the sample interval is allowed to approach zero. This limiting behavior is not true in the general NZSDG since Starr [33] has shown that the so called open-loop feedback solution (where the initial conditions are updated instantaneously along the trajectory) and the closed-loop solution are different for linear problems with no control constraints.

Conditions for Influence Function Continuity Along Singular Surfaces

In higher dimensional problems ($n \geq 3$) singular surfaces which Isaacs calls "universal" surfaces [17] occur frequently when bounded controls appear linearly in the state equation as in the class considered in this chapter. The singular surface is characterized by the fact that trajectories enter the surface from either side then proceed along the surface itself. Letting M denote the singular surface, the situation in two dimensions is depicted in Fig. 1.

From Chapter II we know that the HJB equations are not necessarily defined on switching surfaces (which includes the singular surface M, but the HJB equations are valid on either side of M if we regard the partial derivatives in Eq (3.8) as one sided. If we can show that the

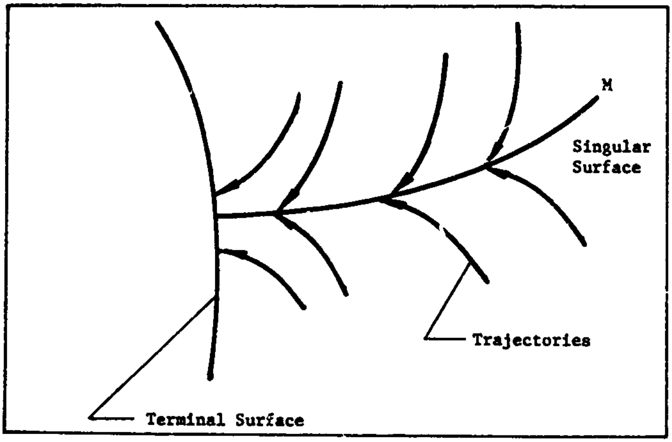

Fig. 1. Singular Surface

partial derivatives, $W^i{}_x$ and $W^i{}_t$ are continuous on M, then we do not need to restrict these partials to be one-sided, and we can conclude that the HJB equations, Eq (3.8), and therefore the influence function equations Eq (3.9) are defined on M.

Hence we establish the following theorem:

Given the NZSDG with the state equation

$$\dot{x} = g\ (x) + \sum_{j=1}^{N} A^j\ (x)\ U^j \qquad x\ (t_o) = x_o \tag{3.1}$$

$$|\ U^j\ | \leq 1$$

with the terminal manifold given by

$$\psi\ [\ x\ (t_f),\ t_f\ ] = 0 \tag{2.3}$$

and cost functions given by

$$J^i = \phi^i [\ x\ (t_f),\ t_f\ ] \tag{3.3}$$

If the following conditions are met then for player k, W^k_x and W^k_t are continuous on M:

i. $g(x)$ and $A^j(x)$, j = 1, . . . , N, are continuous vector functions of x and linearly independent for all x.

ii. M is a singular surface of the control U^i where M can be entered from either side by employing $U^{i*} = +1$ on one side of M or $U^{i*} = -1$ on the other side of M. Let M^+ be the side of M corresponding to $U^{i*} = +1$ and M^- be the side of M corresponding to $U^{i*} = -1$.

iii. The controls U^{j*}, $j \neq i$, are constant near M.

iv. On M the term $W^k_x A^i$ (where W^k_x is considered to be a one sided derivative) is identically zero,

$$W^k_x A^i = 0 \qquad k \neq i \tag{3.14}$$

v. The inner product B (x) A^i is not a constant over any nonzero time interval; B (x) is defined below.

<u>Proof</u>

The HJB equations for player k are

$$W^k_t = -W^k_x g(x) - \sum_{j=1}^{N} W^k_x A^j U^{j*} \qquad k = 1, \ldots, N \tag{3.8}$$

Eq (3.8) is defined on either side of M but not on M itself, therefore the partial derivatives W^k_x and W^k_t must be considered as one-sided on M^+ and M^-. Designate W^{k+}_x and W^{k+}_t as one sided partial derivatives on M^+ and W^{k-}_x and W^{k-}_t as one-sided partial derivatives on M^-.

On M, which is a singular surface for player i, the switching function $W^i_x A^i$ is identically zero (W^i_x is defined on M [27, 28])

$$W^i_x A^i = 0 \tag{3.15}$$

From Eq (3.8) and condition iv. above we can write the equations

$$W^{k+}_x A^{i+} = W^{k-}_x A^{i-} = 0 \tag{3.16}$$

$$W^{k+}_t + W^{k+}_x \left[g(x)^+ + \sum_{j=1}^{N} A^{j+} U^{j*+} \right] = W^{k-}_t + W^{k-}_x \left[g(x)^- + \sum_{j=1}^{N} A^{j-} U^{j*-} \right] = 0$$

Since $g(x)$ and $A^j(x)$, $j = 1, \ldots, N$, are continuous in x, and U^{j*}, $j \neq i$, is constant in the neighborhood of M under consideration, by continuity arguments

$$A^{j+} = A^{j-} = A^j \tag{3.17}$$

and

$$\begin{aligned} B(x) &\equiv g(x)^+ + \sum_{j=1}^{N} A^{j+} U^{j*+} \qquad j \neq i \\ &= g(x)^- + \sum_{j=1}^{N} A^{j-} U^{j*-} \qquad j \neq i \\ &= g(x) + \sum_{j=1}^{N} A^{j} U^{j*} \qquad j \neq i \end{aligned} \tag{3.18}$$

$B(x)$ is the time derivative of the equilibrium state vector except the component containing U^i. By assumption i. $B(x)$ and $A^j(x)$, $j = 1, \ldots, N$, are linearly independent vectors.

From Eqs (3.16), (3.17), and (3.18) we can now write the equations

$$(W^{k+}_x - W^{k-}_x) A^i = 0 \qquad k \neq i \tag{3.19}$$

$$-(W^{k+}_t - W^{k-}_t) = (W^{k+}_x - W^{k-}_x) B(x) \tag{3.20}$$

Now W^{k+}_t and W^{k-}_t are constants on an equilibrium trajectory along M since the state equation is autonomous and the problem has terminal cost functions; hence Eq (3.20) becomes

$$C^k = (W^{k+}_x - W^{k-}_x) B(x) \qquad k \neq i \tag{3.21}$$

Returning to Eq (3.19) one of two conditions exist; either (a), $(W^{k+}_x - W^{k-}_x) = 0$, or (b), $(W^{k+}_x - W^{k-}_x)$ is orthogonal to A^i. Condition (a) together with Eq (3.20) imply $W^{k+}_x = W^{k-}_x$ and $W^{k+}_t = W^{k-}_t$. We will

show by contradiction that only (a) is possible. To obtain a contradiction to (b) suppose (b) is true. Then the vector $(W^{k+}{}_x - W^{k-}{}_x)$ lies in a hyperplane orthogonal to A^i at each x along a trajectory. In Eq (3.21) on the other hand, the vector $(W^{k+}{}_x - W^{k-}{}_x)$ lies in a cone about the vector B (x) for each x along the trajectory. Inspection of Eqs (3.19) and (3.20) reveals that in order for both equations to be satisfied for each x along a trajectory the inner product $\langle B(x), A^i(x)\rangle$ must be constant which is impossible by assumption v. (Since B (x) is the time derivative of most of the state vector while A^i (x) is only the control coefficient for the ith player, assumption v. is a very reasonable one and is not likely to be violated in any realistic problem.) Thus, the contradiction to (b) is established.

The conditions

$$W^{k+}{}_x = W^{k-}{}_x = W^k{}_x$$

$$W^{k+}{}_t = W^{k-}{}_t = W^k{}_t$$

imply that $W^k{}_x$ and $W^k{}_t$ are continuous. Furthermore, because of this continuity, the HJB equations and thus the influence function equations are defined on the singular surface. The advantages of this continuity were enumerated in the introduction to this chapter, and of course the validity of the HJB equations and influence function equations is essential to obtain singular trajectories. Application of the theorem will be illustrated in the problems of Chapters IV and V and in Chapter V it is shown that there are problems for which the theorem does not hold.

IV. Interceptor-Penetrator Problems

In this chapter we use a simple model to illustrate the application of the theory presented in Chapter II and III and also the salient features of the intercept problem treated as a NZSDG. The state equation used to model the vehicles' motion is the same type as in Eq (3.1) of Chapter III so that the control laws are the "bang-bang" variety with the possibility of singular controls. The theorem of Chapter III which tests the continuity of the influence functions on the singular surfaces is applicable, and the influence functions will be shown to be continuous on the singular surfaces. Two problems are considered: (1) a two player intercept problem with one player, the attacker "a", attempting to reach a fixed target and the other player, the defender "d", attempting to intercept a. Termination of the game is achieved when d achieves a separation distance from a of some arbitrary length, say ℓ. We will assume throughout that termination always takes place although an important part of any practical problem is assuring oneself that termination can indeed be accomplished. The defender's goal is interception in minimum time while the attacker's goal is minimization of his final range to the target; (2) a three player problem which is the same as (1) except that another defending player c is added. The termination criteria is taken to be when either c or d intercepts a. General solutions are discussed, and the totally singular solution is solved.[1]

1. In considering the singular surfaces we will be concerned only with the singular surfaces which intersect the terminal manifold since these appear to be the only significant singular surfaces in the practical intercept problem. Intermediate singular arcs can occur as in Isaacs' homocidal chaueffer problem [17], however these intermediate arcs seem to occur when the players are initially inside each other's turning radius—a case which will not be considered in this dissertation.

The theorem of Chapter III is shown to be satisfied so that the influence functions are continuous on the singular surfaces. This continuity allows the totally singular problems to be solved numerically. Numerical solutions of both the two and three player totally singular problems are carried out to illustrate application.

Problem Formulation - Two Players

The equations of motion for the two players are

$$
\begin{aligned}
\dot{x}^d &= V^d \cos \gamma^d \\
\dot{y}^d &= V^d \sin \gamma^d \\
\dot{\gamma}^d &= c^d\, u \\
\dot{x}^a &= V^a \cos \gamma^a \\
\dot{y}^a &= V^a \sin \gamma^a \\
\dot{\gamma} &= c^a\, v
\end{aligned}
\tag{4.1}
$$

Define the state vector x to be

$$x^T = (x^d,\ y^d,\ \gamma^d,\ x^a,\ y^a,\ \gamma^a) \tag{4.2}$$

Fig. 2 shows the inertial position and velocity of the two players and the inertial position of the attacker's target.

The termination criteria needed to end the game is taken to be the satisfaction of the equation

$$\Psi\ [x\ (t_f),\ t_f] = 1/2\ [\ (x^d - x^a)^2 + (y^d - y^a)^2 - \ell^2]_{t=t_f} = 0 \tag{4.3}$$

where ℓ is arbitrary (In a practical problem ℓ could represent maximum launch range for the defender's ordnance. See Fig. 4). The part of the intercept problem which makes it a nonzero sum game is the difference in the goals of the two players. The defender's goal is minimization of the game termination time t_f while the attacker's goal is minimization of the final range to his target $(x_T,\ y_T)$. The cost functions for the

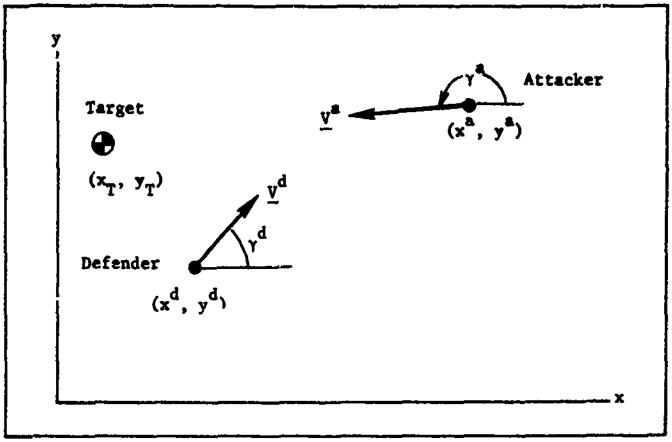

Fig. 2. Inertial Position and Velocity of the Attacker and Defender

defender and attacker are respectively

$$J^d = \phi^d [x(t_f), t_f] = t_f \tag{4.4}$$

$$J^a = \phi^a [x(t_f), t_f] = 1/2 \, [(x^a - x_T)^2 + (y^a - y_T)^2]_{t=t_f}$$

We can now proceed with the formulation of the HJB equations, the influence function equations and the TPBVP. The Hamiltonian functions are

$$\begin{aligned} H^s = {} & \lambda^s_{x^d} V^d \cos\gamma^d + \lambda^s_{y^d} V^d \sin\gamma^d + \lambda^s_{\gamma^d} c^d u \\ & + \lambda^s_{x^a} V^a \cos\gamma^a + \lambda^s_{y^a} V^a \sin\gamma^a + \lambda^s_{\gamma^a} c^a v \end{aligned} \tag{4.5}$$

where the superscript "s" is to be replaced by either "a" for the attacker or "d" for the defender. The systems of HJB equations for the two players is

$$\begin{aligned} W^d_t &= -\min_u H^d \qquad & W^d &= \phi^d [x(t_f), t_f] \\ W^a_t &= -\min_v H^a \qquad & W^a &= \phi^a [x(t_f), t_f] \end{aligned} \tag{4.6}$$

where λ^s in Eq (4.5) is replaced with the partial derivatives W^s_x. The influence function equations for the two players are

$$
\begin{aligned}
\dot{\lambda}^s_{x^d} &= 0 \\
\dot{\lambda}^s_{y^d} &= 0 \\
\dot{\lambda}^s_{\gamma^d} &= V^d \left(\lambda^s_{x^d} \sin \gamma^d - \lambda^s_{y^d} \cos \gamma^d\right) \\
\dot{\lambda}^s_{x^a} &= 0 \\
\dot{\lambda}^s_{y^a} &= 0 \\
\dot{\lambda}^s_{\gamma^a} &= V^a \left(\lambda^s_{x^a} \sin \gamma^a - \lambda^s_{y^a} \cos \gamma^a\right)
\end{aligned}
\tag{4.7}
$$

When referring to the influence function equations for the attacker or defender, replace the superscript "s" with "a" or "d" respectively. Note the absence of either control in the influence function equations Eq (4.7), (4.8), and (4.9) which means that these equations are coupled only at the terminal manifold by the transversality conditions. The transversality conditions are given by Eq (2.25) and in component form are

$$
\lambda^d(t_f) = -1/\,\dot{\Psi}\,(t_f) \begin{bmatrix} -(x^a - x^d) \\ -(y^a - y^d) \\ 0 \\ (x^a - x^d) \\ (y^a - y^d) \\ 0 \end{bmatrix}_{t=t_f}
\tag{4.8}
$$

$$\lambda^a(t_f) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ (x^a - x_T) \\ (y^a - y_T) \\ 0 \end{bmatrix}_{t=t_f} - \dot{\phi}^a(t_f)/\dot{\Psi}\,(t_f) \begin{bmatrix} -(x^a - x^d) \\ -(y^a - y^d) \\ 0 \\ (x^a - x^d) \\ (y^a - y^d) \\ 0 \end{bmatrix}_{t=t_f}$$

where

$$\dot{\Psi} = (x^a - x^d)\ (V^a \cos \gamma^a - V^d \cos \gamma^d) + (y^a - y^d) \tag{4.9}$$

$$(V^a \sin \gamma^a - V^d \sin \gamma^d)$$

and

$$\dot{\phi}^a(t_f) = [\ (x^a - x_T)\ V^a \cos \gamma^a + (y^a - y_T)\ V^a \sin \gamma^a]_{t=t_f} \tag{4.10}$$

The minimizing controls u^* and v^* are

$$u^* = -\ \mathrm{sgn}\ \lambda^d_{\gamma d}$$

$$v^* = -\ \mathrm{sgn}\ \lambda^a_{\gamma a} \tag{4.11}$$

except on singular manifolds where $u^* = 0$ and/or $v^* = 0$ are employed. Appendix D contains the derivation of these singular controls. The TPBVP whose solution is equivalent to solving the NZSDG posed consists of the state equations Eq (4.1) with the controls u^* and v^* substituted from Eq (4.11) (or singular controls if appropriate), and the influence function equations for both players as given by Eq (4.7).

Control Laws

The equations in Eq (4.11) are bang-bang control laws, so that the control pair is always one of the following:

1. $u^* = \pm 1 \qquad v^* = \pm 1$
2. $u^* = 0 \qquad v^* = \pm 1$

3. $u^* = \pm 1 \quad v^* = 0$

4. $u^* = 0 \quad v^* = 0$

As stated in Chapter III, a closed-loop control law requires the solution for all the switching surfaces. The location of the state relative to these surfaces then completely specifies the nonsingular controls u^* and v^*. Trajectories lying along a singular surface have controls from the control pairs 2., 3., or 4. above. These singular trajectories are discussed below.

Singular Surfaces

In Appendix D the Legendre-Clebsh necessary conditions are applied to derive the admissible singular control candidates $u^* = 0$ and/or $v^* = 0$. Additional requirements for a singular control u^* are that on a singular trajectory for d

$$\lambda^d_{\gamma^d} = \dot{\lambda}^d_{\gamma^d} = \ddot{\lambda}^d_{\gamma^d} = \ldots = 0 \tag{4.12}$$

$$\lambda^d_{x^d} \cos \gamma^d + \lambda^d_{y^d} \sin \gamma^d \neq 0 \tag{4.13}$$

For a terminal singular trajectory on which u^* is singular, using Eqs (4.7), (4.8), (4.9) and (4.12) it can be shown that

$$\begin{aligned}
\sin \gamma^d &= [- \lambda^d_{y^d} / (\lambda^{d^2}_{x^d} + \lambda^{d^2}_{y^d})^{1/2}]_{t=t_f} \\
\cos \gamma^d &= [- \lambda^d_{x^d} / (\lambda^{d^2}_{x^d} + \lambda^{d^2}_{y^d})^{1/2}]_{t=t_f} \\
\tan \gamma^d &= [\lambda^d_{y^d} / \lambda^d_{x^d}]_{t=t_f} = [(y^a - y^d) / (x^a - x^d)]_{t=t_f}
\end{aligned} \tag{4.14}$$

By substituting the expressions for $\sin \gamma^d$ and $\cos \gamma^d$ from Eq (4.14) into the inequality in Eq (4.13) we see that the inequality is strictly satisfied.

Eq (4.14) implies that the flight path angle γ^d is such that the velocity vector for the defender lies on his line of sight to the

attacker at termination. The situation is depicted in Fig. 3. Similarly, in order for v^* to be singular Eqs (4.12) and (4.13) with "a" replacing "d" must hold along the singular trajectory. For a terminal singular trajectory on which v^* is singular

$$\tan \gamma^a = [\ \lambda^a{}_{y^a} / \lambda^a{}_{x^a}\ ]_{t=t_f} \tag{4.15}$$

$$= \left[\frac{(y^a - y_T) - (\dot{\phi}^a / \dot{\Upsilon})\ (y^a - y^d)}{(x^a - x_T) - (\dot{\phi}^a / \dot{\Upsilon})\ (x^a - x^d)} \right]_{t=t_f}$$

where $\dot{\phi}^a$ and $\dot{\Upsilon}$ are given by Eqs (4.9) and (4.10). Eq (4.15) implies that the singular trajectory for the attacker has a constant flight path angle which depends upon the relative position of both players at termination and the target. Typical singular trajectories are depicted in Fig. 4.

Continuity of the Influence Functions

The requirements of the continuity theorem in Chapter III are met (by inspection) with the possible exception of condition iv. which must be checked. If condition iv. is met then the influence functions are continuous at the junctions with and on the singular surfaces in this problem. We now show that condition iv. is satisfied. Assume player d switches from a nonsingular to a singular control.

Since $\lambda^a{}_{\gamma d}$ and $W^a{}_{\gamma d}$ are identical on an equilibrium trajectory, condition iv. Eq (3.14) applied to this problem requires that

$$\lambda^a{}_{\gamma d} = 0 \tag{4.16}$$

along every trajectory in d's singular surface. Because $\lambda^a{}_{\gamma d}$ and $W^a{}_{\gamma d}$ are identical on an equilibrium trajectory, $\lambda^a{}_{\gamma d}$ can be regarded as the sensitivity of player a's cost function to a variation in the flight path angle γ^d of player d. From Eqs (3.14), (4.7) and (4.16) we

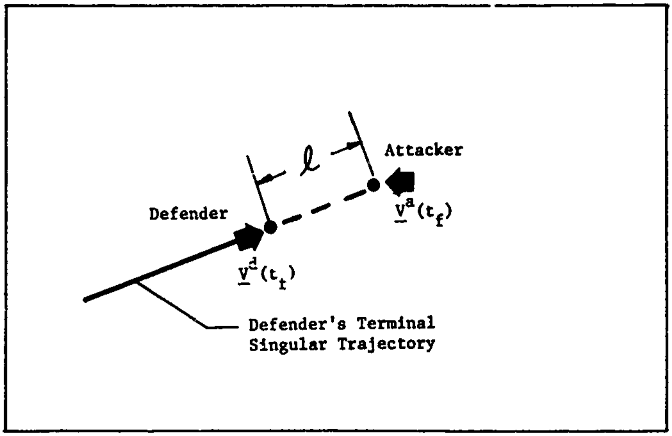

Fig. 3. Defender Singular Arc

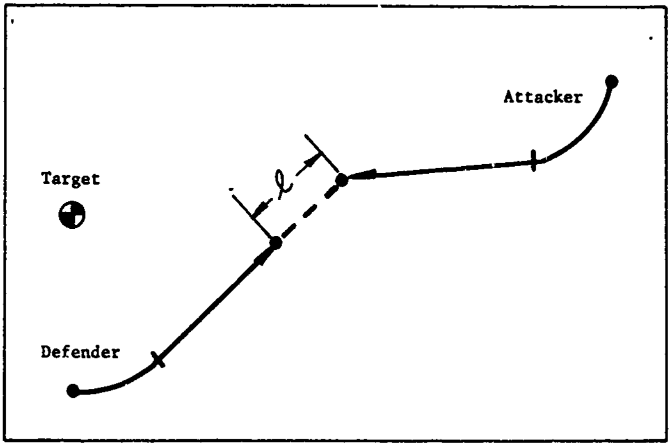

Fig. 4. Typical Singular Trajectories for the Attacker and Defender

can now develop a relationship among the influence function variables of both player a and player d which must hold on d's singular surface. It is then easy to demonstrate that the relationship holds which in turn implies the continuity of the influence functions.

From Eq (3.14) all time derivatives of $\lambda^a_{\gamma d}$ are also zero; therefore from Eq (4.7) the equation $\dot{\lambda}^a_{\gamma d} = 0$ requires that

$$\lambda^a_{xd} \sin \gamma^d - \lambda^a_{yd} \cos \gamma^d = 0 \tag{4.17}$$

or

$$\tan \gamma^d = \lambda^a_{yd} / \lambda^a_{xd}$$

Eq (4.17) implies that the flight path angle of player d can be specified in terms of player a's influence functions. Since $\tan \gamma^d$ is a constant on d's terminal singular trajectory and is specified by Eq (4.14), we can write a relation between player a and player d's influence functions which must hold on the singular surface,

$$\lambda^a_{yd} / \lambda^a_{xd} = (\lambda^d_{yd} / \lambda^d_{xd})_{t=t_f} \tag{4.18}$$

Since the terms in Eq (4.18) are constants, we need only to assure ourselves that the equation and thus condition iv., is satisfied at t_f. This verification can be made from the transversality conditions Eq (4.8)

$$\begin{aligned}(\lambda^a_{yd} / \lambda^a_{xd})_{t=t_f} &= [(y^a - y^d) / (x^a - x^d)]_{t=t_f} \\ &= \tan \gamma^d(t_f) \\ &= (\lambda^d_{yd} / \lambda^d_{xd})_{t=t_f}\end{aligned} \tag{4.19}$$

We have now shown that the conditions of the theorem of Chapter III are satisfied for player d's singular surface so that the influence functions are continuous on this singular surface and the HJB equations

and influence function equations are defined on it. A symmetrical argument holds for player a's singular surfaces. It may be possible that a player switches from a nonsingular to a singular control while the other player is already using a singular control. This situation is merely the intersection of the two singular surfaces and no unusual behavior is associated with this intersection.

Totally Singular (Long Range) Problem

When long ranges are involved in the trajectory of each vehicle and the turn radii of the vehicles are very small compared to these ranges, then the equilibrium trajectories are totally singular except for the initial turning segments. In the context of the problem at hand, the trajectories are straight lines. This simplification makes the problem's solution fairly simple.

Suppose the trajectory of each vehicle is totally singular. Then from Eqs (4.14) and (4.15) we have

$$\tan \gamma^d = [(y^a - y^d) / (x^a - x^d)]_{t=t_f} \tag{4.20}$$

$$\tan \gamma^a = \left[\frac{(y^a - y_T) - (\dot{\phi}^a / \dot{\gamma})(y^a - y^d)}{(x^a - x_T) - (\dot{\phi}^a / \dot{\gamma})(x^a - x^d)}\right]_{t=t_f} \tag{4.21}$$

Using the singular controls $u^* = v^* = 0$ and the state equation Eq (4.1) we obtain the algebraic equations

$$\begin{aligned}
x^d(t) &= x^d(t_o) + (t - t_o) V^d \cos \gamma^d \\
y^d(t) &= y^d(t_o) + (t - t_o) V^d \sin \gamma^d \\
x^a(t) &= x^a(t_o) + (t - t_o) V^a \cos \gamma^a \\
y^a(t) &= y^a(t_o) + (t - t_o) V^a \sin \gamma^a
\end{aligned} \tag{4.22}$$

Let $t_o = 0$. For convenience define a point $(\bar{x}^d, \bar{y}^d)$ along the flight path of the defender as

$$\bar{x}^d(t) = x^d(t) + \ell \cos \gamma^d$$
$$\bar{y}^d(t) = y^d(t) + \ell \sin \gamma^d \qquad (4.23)$$

so that Eq (4.22) becomes

$$\bar{x}^d(t) = x^d_o + (V^d\ t + \ell) \cos \gamma^d$$
$$\bar{y}^d(t) = y^d_o + (V^d\ t + \ell) \sin \gamma^d$$
$$x^a(t) = x^a_o + V^a\ t \cos \gamma^a \qquad (4.24)$$
$$y^a(t) = y^a_o + V^a\ t \sin \gamma^a$$

At t_f, the points $(\bar{x}^d, \bar{y}^d)$ and (x^a, y^a) are identical,

$$\bar{x}^d(t_f) = x^a(t_f)$$
$$\bar{y}^d(t_f) = y^a(t_f) \qquad (4.25)$$

Eqs (4.20), (4.21), (4. 4) and (4.25) constitute the TPBVP. Note that the unknowns γ^a, γ^d, and t_f are all determined at the unknown terminal point.

Numerical Method

The following simple algorithm can be used to numerically solve the TPBVP for γ^a, γ^d, and t_f. Let the superscript "o" indicate the initial guess while the superscript "1" indicates the computed value based on the initial guess.

Algorithm

1. Guess γ^{d^o}, $t_f^{\,o}$.
2. Calculate $[x^{a^o}(t_f), y^{d^o}(t_f)]$ and $[x^{a^o}(t_f), y^{a^o}(t_f)]$ from Eqs (4.22), (4.24) and (4.25).

3. Calculate γ^{a^o} from Eq (4.21).

4. Calculate $[x^{a^o}(t_o), y^{a^o}(t_o)]$ from Eq (4.22) using final conditions in 2. and 3. as initial conditions.

5. Determine the error between $[x^{a^o}(t_o), y^{a^o}(t_o)]$ and $[x^a(t_o), y^a(t_o)]$.

6. Correct γ^{d^o} and $t_f{}^o$ by setting $\gamma^{d^1} = \gamma^{d^o} + \Delta\gamma^d$ and $t_f{}^1 = t_f{}^o + \Delta t_f$.

7. Replace γ^{d^o} and $t_f{}^o$ by γ^{d^1} and $t_f{}^1$ and repeat steps 2. through 6. $\Delta\gamma^d$ and Δt_f may be determined by a number of schemes, many of which may be implemented by using the optimization program AESOP (Automated Engineering and Scientific Optimization Program) [12, 13]. Continue the iterations until the error in Step 5. has been reduced to a suitable bound.

The above algorithm is implemented to illustrate a numerical solution.

Initial Conditions			Initial Guess
$x^d(0) = 0$	$x^a(0) = 2000$	$V^d = 1000$	$\gamma^{d^o} = 1$ radian
$y^d(0) = 0$	$y^a(0) = 1000$	$V^a = 500$	$t_f{}^o = 1.5$
$x_T = 0$	$y_T = 500$	$\ell = 25$	

Program AESOP [13] was used with an IBM 7094 digital computer to obtain the corrections $\Delta\gamma^d$, Δt_f in the algorithm. $\Delta\gamma^d$ and Δt_f were selected to minimize the error in Step 5.,

$$(\text{error}) = \{[x^a(t_o) - x^{a^o}(t_o)]^2 + [y^a(t_o) - y^{a^o}(t_o)]^2\}^{1/2}$$

Using three minutes of machine time, the error was reduced from the initial guess error of 114 to a final error of .063. The final error

represents a distance difference of .063 between the calculated initial point $x^{a^o}(t_o)$, $y^{a^o}(t_o)$ and the actual initial point $x^a(t_o)$, $y^a(t_o)$. The computed values of γ^d, γ^a, and t_f corresponding to the final error are $\gamma^d = 54.6^o$, $\gamma^a = 261.6^o$, and $t_f = 1.51$.

In an actual intercept problem the algorithm could be employed to provide a sampled data feedback control law provided the sample intervals are small. In this control law the state is sampled at discreet times and the sampled state is used as a new initial condition. Employing the algorithm for each state sample updates the flight path angles γ^d and γ^a and the time to intercept, t_f.

If one of the players, say a, employs a nonequilibrium control law, this feedback control law for player d insures a better final cost for player d. It should be noted that the smaller the sample interval is the quicker will be player d's reaction to nonequilibrium play by player a.

Three Player Formulation

Three player differential games have been solved for pursuit-evasion problems [4], however, the games are posed as zero-sum games with the third player introduced by some artifice such as a constraint or a specified guidance law rather than as an independent minimizing player. In this section an additional independent player will be added to the problem of two players already presented in this chapter. The selection of the third player's cost function dictates the degree of his cooperation with the original defender of the two player game.

Consider the two player problem in this chapter, and add to the state equation Eq (4.1) the equations of motion for the third player whom we shall call the cooperating player "c",

$$\dot{x}^c = V^c \cos \gamma^c$$

$$\dot{y}^c = V^c \sin \gamma^c \qquad (4.26)$$

$$\dot{\gamma}^c = c^c w \qquad |w| \leq 1$$

The state vector is now

$$x^T = (x^d, y^d, \gamma^d, x^c, y^c, \gamma^c, x^a, y^a, \gamma^a) \qquad (4.27)$$

Suppose that the goal of the cooperating player is identical to the original defender's goal of interception of the attacker in minimum time,

$$J^c = \phi^c = t_f \qquad (4.28)$$

Depending on the relative positions of the players and assuming termination can take place, one or the other of the defenders achieves intercept first with the possibility of simultaneous intercept. If there were a termination criteria such as Eq (4.3) for each of the defenders, the termination would be ambiguous. To avoid this problem a new single termination criteria is formulated which includes the termination criteria for each player,

$$\begin{aligned} \Psi [x(t_f), t_f] &= 1/2 \, [\Psi^d] \, [\Psi^c] \qquad (4.29) \\ &= 1/2 \, [(x^d - x^a)^2 + (y^d - y^a)^2 - \ell^2] \\ &\quad [(x^c - x^a)^2 + (y^c - y^a)^2 - k^2] \\ &= 0 \end{aligned}$$

k is the radius of c's capture circle and ℓ is the capture circle radius for d. The game is terminated the first time Eq (4.29) is satisfied.

<u>TPBVP for the Three Player Problem</u>

The set of influence function equations for this problem are the

equations in Eq (4.7) plus the equations for the third player,

$$\dot{\lambda}^s_{xd} = 0$$

$$\dot{\lambda}^s_{yd} = 0$$

$$\dot{\lambda}^s_{\gamma d} = V^d \, (\lambda^s_{xd} \sin \gamma^d - \lambda^s_{yd} \cos \gamma^d)$$

$$\dot{\lambda}^s_{xc} = 0$$

$$\dot{\lambda}^s_{yc} = 0 \qquad (4.30)$$

$$\dot{\lambda}^s_{\gamma c} = V^c \, (\lambda^s_{xc} \sin \gamma^c - \lambda^s_{yc} \cos \gamma^c)$$

$$\dot{\lambda}^s_{xa} = 0$$

$$\dot{\lambda}^s_{ya} = 0$$

$$\dot{\lambda}^s_{\gamma a} = V^a \, (\lambda^s_{xa} \sin \gamma^a - \lambda^s_{ya} \cos \gamma^a)$$

where s is one of the set (d, c, a) which refers to the defender, cooperating defender and attacker respectively, thus Eq (4.30) contains 27 component equations.

The transversality conditions are

$$\lambda^s(t_f) = [\phi^s_x - (\dot{\phi}^s / \dot{\psi}) \, \psi_x]_{t=t_f} \qquad (2.25)$$

Eq (2.25) is expressed in component form in Appendix E for this problem.

The bang-bang control laws for the original defender and attacker remain unchanged while the control law for the cooperating defender has the same form,

$$w^* = - \operatorname{sgn} \lambda^c_{\gamma c} \qquad (4.31)$$

The singular control for each player is unchanged,

$$u^* = 0 \qquad v^* = 0 \qquad w^* = 0 \qquad (4.32)$$

The necessary conditions for player d to have a singular control are

$$\lambda^d_{\gamma^d} = 0 \qquad (4.33)$$

$$\lambda^d_{x^d} \sin \gamma^d + \lambda^d_{y^d} \cos \gamma^d \leq 0 \qquad (4.34)$$

and if the problem terminates with a singular control $u^* = 0$, then on the singular trajectory

$$\sin \gamma^d = [- \lambda^d_{y^d} / ({\lambda^{d}}^2_{x^d} + {\lambda^{d}}^2_{y^d})^{1/2}]_{t=t_f}$$

$$\cos \gamma^d = [- \lambda^d_{x^d} / ({\lambda^{d}}^2_{x^d} + {\lambda^{d}}^2_{y^d})^{1/2}]_{t=t_f} \qquad (4.35)$$

or

$$\tan \gamma^d = [(y^a - y^d) / (x^a - x^d)]_{t=t_f}$$

For player c to have a singular control $w^* = 0$,

$$\lambda^c_{\gamma^c} = 0 \qquad (4.36)$$

$$\lambda^c_{x^c} \sin \gamma^c + \lambda^c_{y^c} \cos \gamma^c \leq 0 \qquad (4.37)$$

and if the problem terminates in a singular control $w^* = 0$, then on the singular trajectory

$$\sin \gamma^c = [- \lambda^c_{y^c} / ({\lambda^{c}}^2_{x^c} + {\lambda^{c}}^2_{y^c})^{1/2}]_{t=t_f}$$

$$\cos \gamma^c = [- \lambda^c_{x^c} / ({\lambda^{c}}^2_{x^c} + {\lambda^{c}}^2_{y^c})^{1/2}]_{t=t_f} \qquad (4.38)$$

or

$$\tan \gamma^c = [(y^a - y^c) / (x^a - x^c)]_{t=t_f}$$

Finally, for v^* to be a singular control

$$\lambda^a_{\gamma^a} = 0 \qquad (4.39)$$

$$\lambda^{a}_{x^a} \sin \gamma^a + \lambda^{a}_{y^a} \cos \gamma^a \leq 0 \tag{4.40}$$

and if the problem terminates in a singular control $v^* = 0$, then on the singular trajectory

$$\tan \gamma^a = \left[\frac{(y^a - y_T) - (\dot{\phi}^a / \dot{\Psi})\ \partial\Psi/\partial y^a}{(x^a - x_T) - (\dot{\phi}^a / \dot{\Psi})\ \partial\Psi/\partial x^a} \right]_{t=t_f} \tag{4.41}$$

The inequalities of Eqs (4.34), (4.37) and (4.40) are shown to hold with strict inequality in the same manner as in Eq (4.13).

Because of the termination criteria Eq (4.29), one or the other or both of the defending players will make the intercept and cause termination. Recalling the termination criteria function

$$\Psi\ [x\ (t_f),\ t_f] = 1/2\ [\ \Psi^d]\ [\ \Psi^c] = 0 \tag{4.29}$$

$\Psi = 0$ implies the following possibilities;

$$\begin{aligned} &\text{(a)} \quad \Psi^d = 0 && \text{(player d intercepts first)} \\ &\text{(b)} \quad \Psi^c = 0 && \text{(player c intercepts first)} \\ &\text{(c)} \quad \Psi^d = \Psi^c = 0 && \text{(simultaneous intercept)} \end{aligned} \tag{4.42}$$

If (a) holds in a problem, then examination of the transversality conditions in Appendix E reveals that the trajectory of the attacker does not depend on the position of the nonintercepting player c. In other words, the attacker and intercepting player d have the same trajectories as in the two player game where the nonintercepting player c is excised from the problem. The nonintercepting player's trajectory is determined by Eq (4.38) which implies that if a singular arc is attained by player c, at termination his line of sight to the attacker is coincident with his velocity vector. The situation is depicted in Fig. 5. A similar statement holds for case (b).

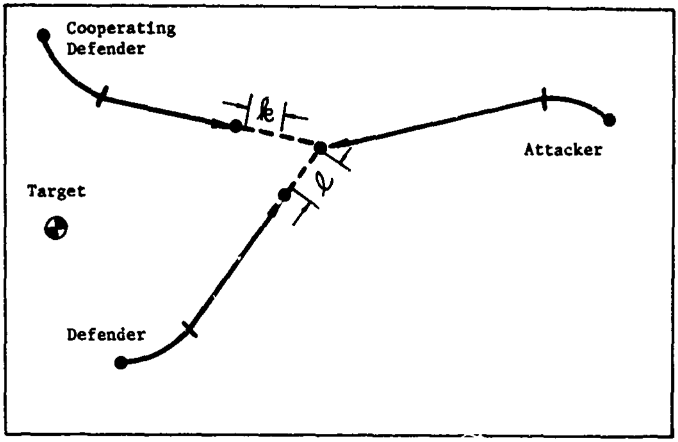

Fig. 5. Typical Three Player Trajectory

Consider case (c). In the two player problem, if a third player (the cooperating defender) is added to the problem and if his capability permits, the three player problem solution will differ from the solutions for cases (a) and (b) where the attacker and one of the defenders play as if the second defender were not present. When the cooperating player can influence the solution, but not force case (b) then case (c) holds and the intercept is made simultaneously by both defenders. Examination of the transversality conditions for the attacker in Appendix E, reveals that for case (c), $\lambda^{a}{}_{x^a}(t_f)$ and $\lambda^{a}{}_{y^a}(t_f)$ are indeterminate. Application of l'Hospital's rule to Eq (4.41) removes the indeterminacy. The resulting equation for tan γ^a is

$$\tan \gamma^a(t_f) = \left. \frac{y^a - y_T + \dot{\phi}^a \left[(y^c - y^a) / \dot{\Psi}^c + (y^d - y^a) / \dot{\Psi}^d \right]}{x^a - x_T + \dot{\phi}^a \left[(x^c - x^a) / \dot{\Psi}^c + (x^d - x^a) / \dot{\Psi}^d \right]} \right|_{t=t_f} \quad (4.43)$$

The expressions for $\dot{\phi}^a$, $\dot{\Psi}^c$ and $\dot{\Psi}^d$ are found in Appendix E.

The theorem of Chapter III is satisfied on the terminal singular surfaces of all three players. This can be verified in the same manner as in the earlier two player problem of this chapter.

We can now proceed to develop the totally singular solution.

Totally Singular Three Player Problem

As in the two player problem, if the turn radii are assumed very small compared to the total range of each player, then the problem reduces to one in which the trajectories are totally singular - in this problem, straight lines in the plane. Following the format of the two player totally singular problem, the equations defining the flight path angles are given by Eqs (4.35), (4.38) and (4.41) repeated here for reference,

$$\tan \gamma^d = [(y^a - y^d) \; / \; (x^a - x^d)]_{t=t_f} \tag{4.35}$$

$$\tan \gamma^c = [(y^a - y^c) \; / \; (x^a - x^c)]_{t=t_f} \tag{4.38}$$

$$\tan \gamma^a = \left. \frac{(y^a - y_T) - (\dot{\phi}^a / \dot{\Psi}) \; \partial\Psi/\partial y^a}{(x^a - x_T) - (\dot{\phi}^a / \dot{\Psi}) \; \partial\Psi/\partial x^a} \right|_{t=t_f} \tag{4.41}$$

If a simultaneous intercept occurs, Eq (4.41) is replaced by Eq (4.43). The state equation is integrated using the singular controls u^*, v^*, and w^* to give (assume $t_o = 0$)

$$\begin{aligned} x^d(t) &= x^d_o + V^d \, t \cos \gamma^d \\ y^d(t) &= y^d_o + V^d \, t \sin \gamma^d \\ x^c(t) &= x^c_o + V^c \, t \cos \gamma^c \\ y^c(t) &= y^c_o + V^c \, t \sin \gamma^c \end{aligned} \tag{4.44}$$

$$x^a(t) = x^a_o + V^a\ t \cos \gamma^a$$

$$y^a(t) = y^a_o + V^a\ t \sin \gamma^a$$

Define $\bar{x}^d$, $\bar{y}^d$, $\bar{x}^c$, $\bar{y}^c$ by the following equations,

$$\begin{aligned}
\bar{x}^d(t) &= x^d_o + (\ V^d\ t + \ell) \cos \gamma^d \\
\bar{y}^d(t) &= y^d_o + (\ V^d\ t + \ell) \sin \gamma^d \\
\bar{x}^c(t) &= x^c_o + (\ V^c\ t + k) \cos \gamma^c \\
\bar{y}^c(t) &= y^c_o + (\ V^c\ t + k) \sin \gamma^c
\end{aligned} \tag{4.45}$$

When $t \to t_f$, one of three conditions holds; either

(a) $\bar{x}^d(t_f) = x^a(t_f)$ and $\bar{y}^d(t_f) = y^a(t_f)$

or (b) $\bar{x}^c(t_f) = x^a(t_f)$ and $\bar{y}^c(t_f) = y^a(t_f)$ (4.46)

or (c) both (a) and (b).

Case (a) corresponds to intercept by player d, case (b) corresponds to intercept by player c and case (c) corresponds to simultaneous intercept. The TPBVP consists of finding the angles γ^a, γ^c, and γ^d which are defined at the unknown terminal point by Eqs (4.35), (4.38) and (4.41) such that the conditions in Eq (4.46) are met.

Numerical Method

An algorithm which can be used to solve numerically for the unknown angles in the TPBVP is based on the three intercept conditions of Eq (4.42).

Algorithm

1. Assume intercept by player c.
2. Solve the two player (c and a) NZSDG using the algorithm for the two player totally singular problem. This provides an intercept time for player c.

3. Determine whether player d is capable of intercepting first in the solution in 2. by computing player d's time to intercept. If so, the assumption in 1. is not correct, go to 4. If not so, the assumption is correct; determine player d's flight path angle by Eq (4.32).
4. Assume intercept by player d.
5. Solve the two player (d and a) NZSDG using the algorithm for the two player totally singular problem.
6. Determine whether player c is capable of intercepting first in the solution in 5. by computing player c's time to intercept. If so, the assumption in 4. is not correct; go to 7. If not so the assumption is correct; determine player c's flight path angle by Eq (4.38).
7. Reaching this point implies simultaneous intercept. The TPBVP which must be solved consists of satisfying the initial conditions for the problem, the transversality conditions, Eqs (4.35), (4.38) and (4.41) and the termination criteria, Eq (4.42c). Eq (4.43) are the equations of motion for the three vehicles. The following steps yields a numerical solution to the simultaneous intercept situation:

(a) Guess γ^{d^o}

Using Eqs (4.45) and (4.46c), solve for γ^{c^o} and $t_f^{\,o}$ which gives simultaneous intercept for players c and d.

(c) With γ^{d^o}, γ^{c^o}, and $t_f^{\,o}$ calculate $[x^{c^o}(t_f), y^{c^o}(t_f)]$ and $[x^{d^o}(t_f), y^{d^o}(t_f)]$. Compute $[x^{a^o}(t_f), y^{a^o}(t_f)]$ from Eqs (4.24) and (4.25).

(d) Using Eq (4.38) solve for γ^{a^o}.

(e) Using γ^{a^o} and $t_f^{\,o}$, compute $[x^{a^o}(t_o), y^{a^o}(t_o)]$ from Eq (4.44).

(f) Compute the distance error between $[x^{a^o}(t_o), y^{a^o}(t_o)]$ and the initial point for player a, $[x^a(t_o), y^a(t_o)]$.

(g) Define γ^{d^1} by the equation $\gamma^{d^1} = \gamma^{d^o} + \Delta\gamma^d$

(h) Replace γ^{d^o} in (a) by γ^{d^1}.

(i) Repeat steps (b) through (h) until the error in (f) has been reduced to some suitable bound. Program AESOP [13] may be used to obtain the correction $\Delta\gamma^d$ to reduce the error in (f) to nearly zero. The error is computed as in the two player problem.

The algorithm is implemented to illustrate a numerical solution. The parameter values of the two player totally singular problem are used so that the effect of the third player on the trajectories of the original two players may be observed. Three different speeds for the third player c will be used to illustrate the three cases in Eq (4.42).

Initial Conditions

$x^d(0) = 0$	$x^a(0) = 2000$	$V^d = 1000$	Case (A) $V^c = 500$
$y^d(0) = 0$	$y^a(0) = 1000$	$V^a = 500$	Case (B) $V^c = 1000$
$x^c(0) = 0$	$x_T = 0$	$\ell = 25$	Case (C) $V^c = 1500$
$y^c(0) = 1500$	$y_T = 500$	$k = 25$	

The computed values of γ^d, γ^c, γ^a and t_f for cases (A), (B) and (C) are given in Table I.

Table I

Numerical Results for the Three Player Problem

Case	γ^a	γ^c	γ^d	t_f	Target Range
(A)	261.6°	116.1°	54.6°	1.51	1311
(B)	253.6°	120.1°	59.9°	1.47	1317
(C)	247.6°	115.3°	62.1°	1.09	1526

The trajectories of the three player game for cases (A) (B) and (C) are depicted in Fig. 6. Note that in case (A) the trajectories for players a and d are identical to the two player game since the capability of the added player c is not sufficient to cause the attacker to change his flight path angle γ^a. When player c has sufficient speed to affect the solution but not to effect intercept by himself, the attacker changes his flight path angle from that of the two player problem so as to cause a simultaneous intercept. Finally if c's speed is sufficiently large, the attacker plays only against c as in case (C).

Summarizing, in this chapter two simplified intercepter penetrator problems were formulated using NZSDG theory. The continuity theorem of Chapter III was shown to hold on the terminal singular surfaces which implied influence function continuity on the surfaces. This continuity permitted the numerical solution of the totally singular problems.

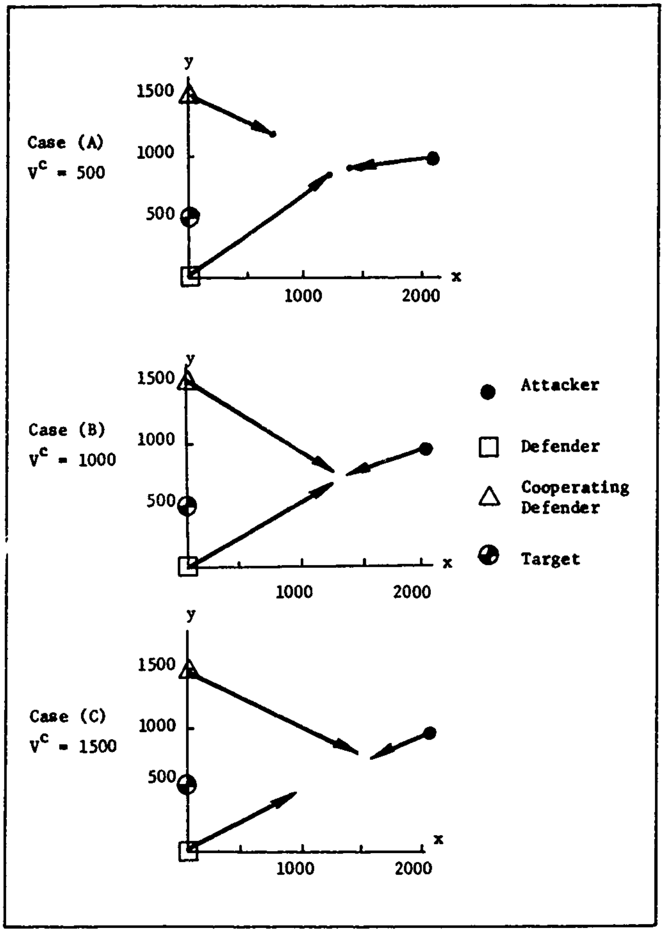

Fig. 6. Three Player Totally Singular Intercept Problem

V. A Pursuit-Evasion Problem

Pursuit-evasion problems have classically been posed as two-player zero sum differential games in which the goals of the players are always opposite. However, in realistic situations it is not always clear that the goals of the players are exactly opposite so that zero sum differential games do not suffice to model the situation. It is this aspect with which this chapter is concerned. Here we solve a two player fixed terminal time NZSDG in which the players have conflicting but not diametrically opposite goals.

Denoting the "pursuer" by p and the "evader" by e, the goal of p is to minimize a function of the final relative range and p's line of sight error while the goal of e is to minimize a different (but not opposite) function of the relative range and p's line of sight error. Relative range and the line of sight error of the pursuer are chosen because of the importance of these functions in pursuit-evasion problems. The dynamic model for the two vehicles is taken from Ref. [5] where it was used in the analysis of a zero sum differential game of the pursuit-evasion type. The problem considered in this chapter will be shown to satisfy the theorem of Chapter III which implies continuity of the influence functions when the controls are discontinuous as the trajectory joins a singular surface.

The objective of this chapter is to characterize the solutions to a NZSDG pursuit-evasion problem; therefore solutions will be completed only to the extent necessary to illustrate the solution behavior.

Problem Formulation

The dynamic model for this problem consists of two vehicles moving in a plane. The equations governing the motion for each vehicle in a vertical plane are

$$
\begin{aligned}
\dot{x} &= V \cos \gamma \\
\dot{y} &= V \sin \gamma \\
\dot{V} &= (T-D)/m - g \sin \gamma \\
\dot{\gamma} &= L/m\,V
\end{aligned}
\tag{5.1}
$$

where x and y are spatial coordinates in the plane of motion, V is the speed, γ the flight path inclination w.r.t. the x axis, T the thrust, and m the mass. The aerodynamic forces are defined by the drag and lift equations $D = 1/2\,\rho\,V^2\,S\,C_D$ and $L = 1/2\,\rho\,V^2\,S\,C_L$.

If the induced drag due to lift is small compared to the total drag, it is possible to approximate the drag D by assuming zero induced drag due to lift. This approximation is especially appropriate when the vehicle speed is great and the acceleration in the lift vector direction is limited because of structural or pilot considerations. To show this, define the load factor n as the ratio of lift force magnitude to vehicle weight,

$$n = \frac{L}{W} \tag{5.2}$$

For a vehicle with a load factor limit n_{max} the maximum lift coefficient $C_{L_{max}}$ is dependent upon the vehicle speed according to the equation

$$C_{L_{max}} = \frac{n_{max}\,Mg}{1/2\,\rho\,V^2\,S} = \frac{K\,(M,\,g,\,\rho,\,S,\,n_{max})}{V^2} \tag{5.3}$$

Assuming n_{max}, m, g, ρ, and S constant for a given flight regime, $C_{L_{max}}$ is inversely proportional to the square of the speed. Assume that the total drag coefficient can be represented by the drag polar equation

$$C_D = C_{D_o} + kC_L^2 \tag{5.4}$$

where C_{D_o} is the zero lift drag coefficient. Then $C_{D_{max}}$ is also a function of vehicle speed

$$C_D = C_{D_o} + k \frac{K^2}{V^4} \tag{5.5}$$

For a typical supersonic fighter aircraft a graph of C_{D_o} vs. Mach number is depicted in Fig. 7 which shows that C_{D_o} approaches an assymptotic value at speeds above M = 2.5. For the same aircraft Fig. 8 shows typical graphs of $C_{L_{max}}$ and $C_{D_{max}}$ vs. V. At V = 2700 ft/sec. the

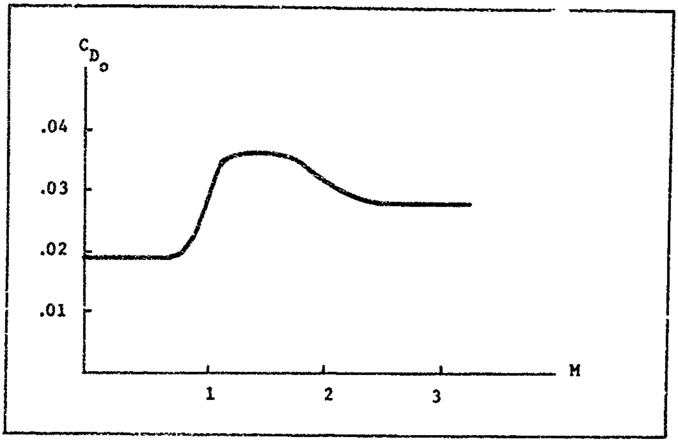

Fig. 7. C_{D_o} vs. M for a Typical Supersonic Fighter

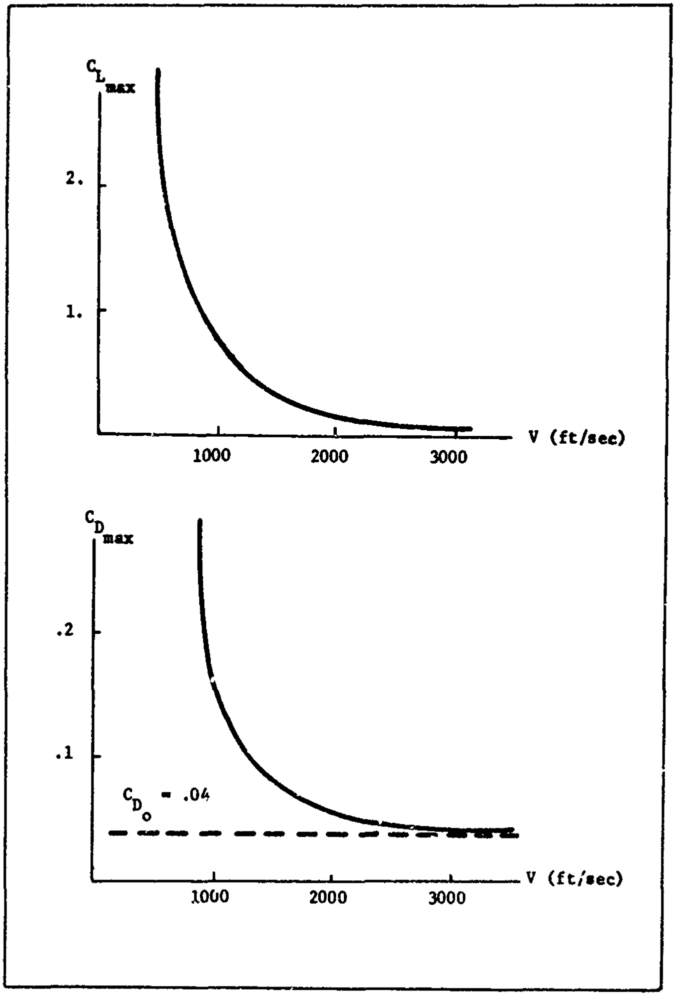

Fig. 8. $C_{L_{max}}$ and $C_{D_{max}}$ vs. V for a Typical Supersonic 'ighter

maximum induced drag factor $kC_{L_{max}}^2$ is less than 5% of the zero lift drag coefficient C_{D_o}. Thus at high speeds the induced drag due to lift can be neglected without significantly affecting the dynamic model.

Using the load factor n as a control variable and neglecting the induced drag factor kC_L^2, the equations of mot.on for a vertical plane become

$$\begin{aligned} \dot{x} &= V \cos \gamma \\ \dot{y} &= V \sin \gamma \\ \dot{V} &= T/m - 1/2\, \rho\, \frac{V^2 S}{m}\, C_{D_o} - (g \sin \gamma) \\ \dot{\gamma} &= \frac{ng}{V} \end{aligned} \tag{5.6}$$

where the control variable n is constrained according to the inequality

$$|n| \le n_{max} \tag{5.7}$$

Since gravity affects both vehicles almost equally, gravitational effects will be neglected. If desired, after the control laws are determined gravity can be replaced in the problem and the trajectories computed to give approximate equilibrium trajectories in the presence of gravity [25]. Considering two vehicles, a pursuer p and evader e the state equations are (neglecting gravitational effects in the V equations)

$$\begin{aligned} \dot{x}^p &= V^p \cos \gamma^p \\ \dot{y}^p &= V^p \sin \gamma^p \\ \dot{V}^p &= T^p/m^p - D_p\, V^{p2}\, C_{D_o}{}^p \\ \dot{\gamma}^p &= n^p g/V^p \\ \dot{x}^e &= V^e \cos \gamma^e \end{aligned} \tag{5.8}$$

$$\dot{y}^e = V^e \sin \gamma^e$$

$$\dot{V}^e = T^e/m^e - D_e V^{e2} C_{D_o}{}^e$$

$$\dot{\gamma}^e = n^e g/V^e$$

where $D_p = 1/2\ \rho\ S^p/m^p$ and $D_e = 1/2\ \rho\ S^e/m^e$

The controls n^p and n^e are constrained according to the inequalities

$$|n^p| \leq n^p{}_{max} \qquad |n^e| \leq n^e{}_{max} \tag{5.9}$$

Fig. 9 shows the coordinates of the two players as well as the "angle-off" angle θ and the range R between the vehicles. The range R and angle off θ are defined by the equations

$$R = [(x^e - x^p)^2 + (y^e - y^p)^2]^{1/2} \tag{5.10}$$

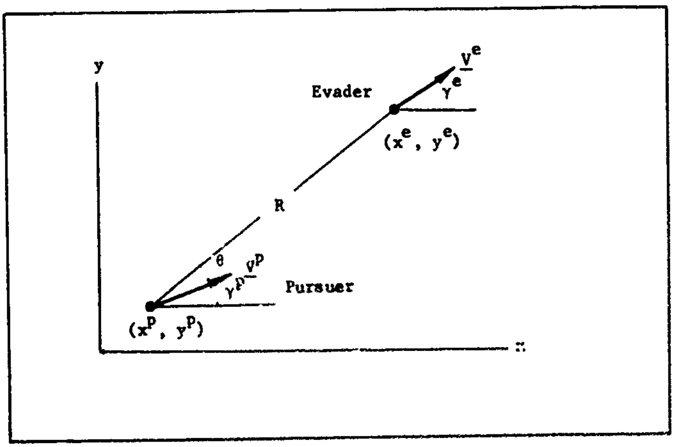

Fig. 9. Inertial Position and Velocity of the Evader and Pursuer

$$\theta = \tan^{-1} \frac{(y^e - y^p)}{x^e - x^p} - \gamma^p \qquad (5.11)$$

The problem is to find the equilibrium controls n^p and n^e for the terminal cost functions

$$J^p = \phi^p [x (t_f), t_f]$$
$$J^e = \phi^e [x (t_f), t_f] \qquad (5.12)$$

subject to the state equations Eq (5.8) and the termination criteria constraint

$$\psi[x (t_f), t_f] = T - t_f = 0 \qquad (5.13)$$

T is a fixed number. To make this problem a NZSDG problem we require

$$J^p \neq - J^e \qquad (5.14)$$

Necessary Conditions

Recall from Chapter II the two different formulations of the equilibrium solution necessary conditions - the HJB partial differential equations Eq (2.14) and the generalized Euler-Lagrange equations Eq (2.20). For this problem the HJB equations are

$$W^s_t = - \min_{n^s} H^s \qquad s = p, e \qquad (5.15)$$

where the boundary conditions for Eq (5.15) are

$$W^s = \phi^s [x (t_f), t_f] \qquad (5.16)$$

The Hamiltonian functions are

$$H^s = \lambda^s_{xp} \ V^p \cos \gamma^p + \lambda^s_{yp} \ V^p \sin \gamma^p + \lambda^s_{Vp} \ (T^p/m^p - D^p V^{p2} \ C_{D_o}) + \lambda^s_{\gamma p} \ n^p g/V^p$$

$$+ \lambda^s_{xe} V^e \cos \gamma^e + \lambda^s_{ye} V^e \sin \gamma^e + \lambda^s_{Ve} (T^e/m^e - D^e V^{e2} C_{D_o}{}^p)$$

$$+ \lambda^s_{\gamma e} n^e g/V^e \tag{5.17}$$

where W^s_x replaces λ^s where H^s is used in Eq (5.15).

The equilibrium controls n^{p*} and n^{e*} are found from the minimization in Eq (5.15), to be

$$n^{p*} = - n^p_{max} \operatorname{sgn} W^p_{\gamma p} \tag{5.18}$$

$$n^{e*} = - n^e_{max} \operatorname{sgn} W^e_{\gamma e} \tag{5.19}$$

Eqs (5.18) and (5.19) hold except where $W^p_{\gamma p} = 0$ and/or $W^e_{\gamma e} = 0$ on a nonzero time interval. In the latter case the possibility of a singular solution exists and must be checked.

The HJB equations Eq (5.15) are the most general form of the necessary conditions; however, solutions to the HJB equations have not been found for this problem so that we are forced to use the more restrictive Euler-Lagrange equations which for this problem consist of the state equation Eq (5.8) and the following influence function equations:

$$\begin{aligned}
&\dot\lambda^s_{xp} = 0 \\
&\dot\lambda^s_{yp} = 0 \\
&\dot\lambda^s_{Vp} = - \lambda^s_{xp} \cos \gamma^p - \lambda^s_{yp} \sin \gamma^p + 2 \lambda^e_{Vp} V^p D_p C_{D_o}{}^p + \lambda^s_{\gamma p} g n^{p*}/V^{p2} \\
&\dot\lambda^s_{\gamma p} = \lambda^s_{xp} V^p \sin \gamma^p - \lambda^s_{yp} V^p \cos \gamma^p \qquad (5.20) \\
&\dot\lambda^s_{xe} = 0 \\
&\dot\lambda^s_{ye} = 0 \\
&\dot\lambda^s_{Ve} = - \lambda^s_{xe} \cos \gamma^e - \lambda^s_{ye} \sin \gamma^e + 2\lambda^s_{Ve} V^e D_e C_{D_o}{}^e + \lambda^s_{\gamma e} g\, n^{e*}/V^{e2}
\end{aligned}$$

$$\dot{\lambda}^{s}_{\gamma^e} = \lambda^{s}_{x^e} V^e \sin \gamma^e - \lambda^{s}_{y^e} V^e \cos \gamma^e$$

where s is replaced by either p or e.

The minimizing equilibrium controls n^{p*} and n^{e*} are obtained from Eqs (5.18) and (5.19) where $W^{p}_{\gamma^p}$ and $W^{e}_{\gamma^e}$ are replaced by $\lambda^{p}_{\gamma^p}$ and $\lambda^{e}_{\gamma^e}$ respectively on an equilibrium trajectory,

$$n^{p*} = - n^{p}_{max} \operatorname{sgn} \lambda^{p}_{\gamma^p} \tag{5.21}$$

$$n^{e*} = - n^{e}_{max} \operatorname{sgn} \lambda^{e}_{\gamma^e} \tag{5.22}$$

When $\lambda^{p}_{\gamma^p}$ and/or $\lambda^{e}_{\gamma^e}$ are identically zero on a nonzero time interval the possibility of a singular solution must be investigated as in Chapter II.

The transversality conditions for the influence function equations Eq (5.20) are

$$\begin{aligned} \lambda^{p} (t_f) &= \phi^{p}_{x} [x (t_f), t_f] \\ \lambda^{e} (t_f) &= \phi^{e}_{x} [x (t_f), t_f] \end{aligned} \tag{5.23}$$

Singular Controls

A detailed study of intermediate singular arcs is beyond the scope of this dissertation hence we will consider only the important class of singular surfaces which intersect the terminal manifold. (In problems with realistic initial conditions the intermediate singular arcs are not likely to occur).

Eqs (5.21) and (5.22) indicate that the controls n^{p*} and n^{e*} are on their respective constraint boundaries except possibly on singular surfaces on which

$$\lambda^{p}_{\gamma^p} = 0 \text{ and/or } \lambda^{e}_{\gamma^e} = 0 \tag{5.24}$$

To characterize the control laws on these singular surfaces, the necessary conditions for singular controls are employed. The resulting control laws are

$$n^{p*} = 0 \tag{5.25}$$

for the pursuer's singular surface and

$$n^{e*} = 0 \tag{5.26}$$

for the evader's singular surface. Appendix D contains the analysis leading to Eqs (5.25) and (5.26).

The control sequence for the pursuer is comprised of control segments from the set ($-n^{p}_{max}$, 0, $+n^{p}_{max}$). Similarly, the control sequence for the evader is comprised of segments from the set ($-n^{e}_{max}$, 0, $+n^{e}_{max}$).

Influence Function Continuity on a Singular Surface

We now show that this problem satisfies the requirements of the theorem in Chapter III for a certain class of cost functions, which implies continuity of the influence functions on the singular surfaces. Conditions i. of the theorem is satisfied which can be verified by examining the state equation Eq (107). By letting U^s be the control and defining

$$n^s = n^s_{max}\, U^s \qquad s = p, e \tag{5.27}$$

$$|U^s| \leq 1 \tag{5.28}$$

we see that condition ii. is satisfied.

Condition iii. is satisfied since on either side of a singular surface, all controls are constant, and we will make the reasonable assumption that the neighborhood under consideration contains no control switching surface except the singular surface. Condition v. is satisfied by inspection of the state equation Eq (5.8). Only condition iv. remains to be shown. Assume p has a singular control. Letting "i" in the theorem be replaced by "p" we must have from Eq (3.15)

$$W^e_{\gamma p} = 0 \tag{5.29}$$

on p's surface. Since singular trajectories proceed along the surface itself, we will replace $W^e_{\gamma p}$ with $\lambda^e_{\gamma p}$. For every trajectory in p's singular surface we thus require

$$\lambda^e_{\gamma p} = 0 \tag{5.30}$$

The condition in Eq (5.30) can be met at the terminal surface provided the transversality condition for $\lambda^e_{\gamma p}\ (t_f)$ is identically zero

$$\lambda^e_{\gamma p}\ (t_f) = \ (\partial\phi^e/\partial\gamma^p)_{t=t_f} = 0 \tag{5.31}$$

The condition in Eq (5.30) can be maintained on trajectories in the singular surface provided

$$\dot{\lambda}^e_{\gamma p}\ (t) = 0 \tag{5.32}$$

on the surface. Eq (5.20) and Eq (5.32) imply that on trajectories in p's terminal singular surface

$$\tan\gamma^p = \lambda^e_{yp}/\lambda^e_{xp} = \lambda^e_{yp}\ (t_f)/\lambda^e_{xp}\ (t_f) \tag{5.33}$$

In Eq (5.33) $\tan\gamma^p$, which is a constant on p's singular surface, is specified in terms of e's influence functions. From the transversality conditions $\tan\gamma^p(t_f) = (\lambda^p_{yp}/\lambda^p_{xp})_{t=t_f}$ so that on p's terminal singular

surface the equality

$$\lambda^{e}_{yp}/\lambda^{e}_{xp} = (\lambda^{p}_{yp}/\lambda^{p}_{xp})_{t=t_f} \tag{5.34}$$

must hold. If Eqs (5.29) through (5.34) hold on p's singular surface, then condition iv. of the theorem in Chapter III is met and we conclude that player e's influence functions are continuous on p's singular surface. A symmetric argument holds for e's singular surface.

Range and Angle-Off Cost Functions

A class of cost functions for which Eqs (5.29) - (5.34) holds is

$$J^{p} = [aR + (1-a)\ \theta^{2}]_{t=t_f} \qquad a\ \varepsilon\ [0,\ 1] \tag{5.35}$$

$$J^{e} = [-bR - (1-b)\ \theta^{2}]_{t=t_f} \qquad b\ \varepsilon\ [0,\ 1]$$

R and θ are defined in Eqs (5.10) and (5.11). To make the problem a NZSDG problem it is only necessary to choose $a \neq b$ in Eq (5.35). (If $a = b$ the game is equivalent to a zero sum differential game.) The cost functions in Eq (5.35) are important in the formulation of pursuit-evasion problems since both the terminal range and the terminal angle-off are significant parameters.

We will now characterize the singular surface for the pursuer, and Eqs (5.30) and (5.32) will be shown to hold for p's singular surface which intersects the terminal manifold. In order to have a singular surface for p which intersects the terminal manifold, from Eqs (5.28) and (5.31) we require

$$\lambda^{p}_{\gamma p}\ (t_f) = 0 \tag{5.36}$$

and

$$\tan \gamma^P (t_f) = (\lambda^P{}_{yP}/\lambda^P{}_{xP})_{t=t_f} \tag{5.37}$$

so that from Eqs (5.35) and (5.36)

$$\lambda^P{}_{\gamma P} (t_f) = [2 (1-a) \theta (-1)]_{t=t_f} = 0 \tag{5.38}$$

which implies (for $a \neq 1$)

$$\theta (t_f) = 0 \tag{5.39}$$

Eq (5.39) implies that the pursuer has his velocity vector on the line of sight to the evader at the final time. This can also be seen from Eq (5.37) which when evaluated becomes

$$\tan \gamma^P (t_f) = [(\partial\phi^P/\partial y^P)/(\partial\phi^P/\partial x^P)]_{t=t_f} \tag{5.40}$$

$$= [(y^e - y^P)/(x^e - x^P)]_{t=t_f}$$

Eq (5.40) implies that at t_f the flight path angle γ^P corresponds with the line of sight angle. Thus, the singular trajectory for the pursuer is a straight line whose direction coincides with the pursuer's velocity vector at t_f. Fig. 10 depicts the situation in which a trajectory for the pursuer contains a terminal singular arc.

Next we wish to show that Eqs (5.30) and (5.32) hold for the cost functions in Eq (5.35), which means that when the pursuer is on his singular surface condition iv. of the theorem in Chapter III is satisfied implying continuity of e's influence functions. Eq (5.30) for this problem becomes

$$\lambda^e{}_{\gamma P} (t_f) = [-2 (1-b) \theta (-1)]_{t=t_f} \tag{5.41}$$

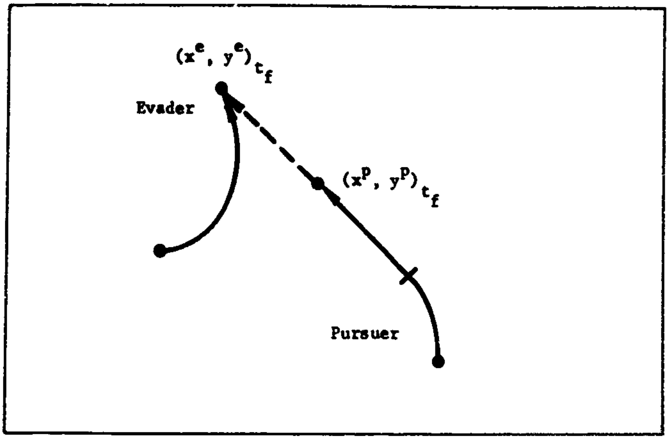

Fig. 10. Singular Terminal Trajectory for the Pursuer

and from Eq (5.39) we conclude

$$\lambda^e_{\gamma p}(t_f) = 0$$

To show Eq (5.32) is true, we have from the transversality conditions

$$(\lambda^e_{yp}/\lambda^e_{xp})_{t=t_f} = [(y^e-y^p)/(x^e-x^p)]_{t=t_f} = \tan\gamma^p(t_f) \qquad (5.42)$$

By substituting from Eq (5.42) into the equation for $\dot{\lambda}^e_{\gamma p}$ Eq (5.20) we see that $\dot{\lambda}^e_{\gamma p}$ is zero on p's singular surface. Thus Eqs (5.30) and (5.32) are satisfied on p's singular surface implying continuity of e's influence functions.

Similar arguments hold for the evaders' singular surface. A situation in which only the evader has a singular terminal arc is depicted in Fig. 11. Fig. 12 depicts a situation in which the singular surfaces of both the pursuer and evader intersect resulting in a tail-chase situation.

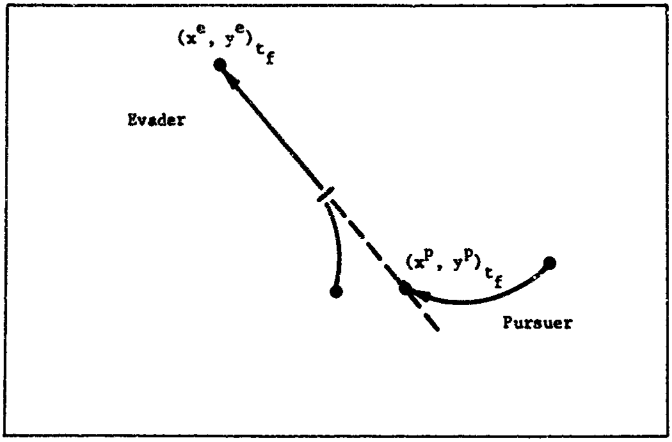

Fig. 11. Singular Terminal Trajectory for the Evader

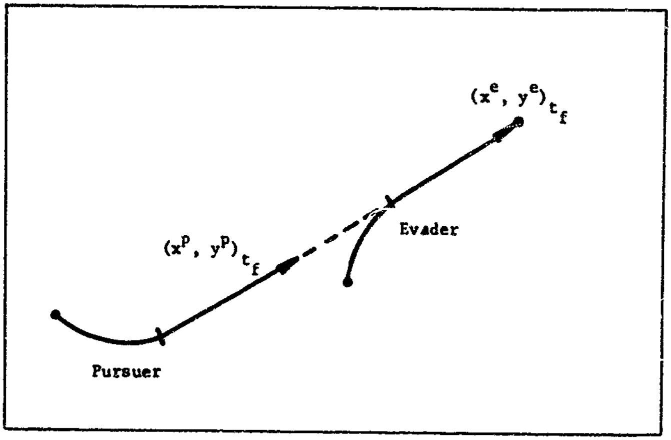

Fig. 12. Typical Singular Trajectory for both Pursuer and Evader

Open and Closed-Loop Control Laws

The control laws in Eqs (5.20), (5.22), and (5.23) are open-loop laws since they require the solution of a two point boundary value problem to obtain them. If, however, the singular surfaces and all other switching surfaces in the problem are known, then closed-loop control laws can be implemented simply by identifying on which side of a particular switching surface the state is located.

In the class of problems considered in this dissertation, the influence function equations are identical for both the open-loop and closed-loop control laws; therefore, closed-loop control laws are theoretically obtainable by solving the open-loop problem at each instant of time along a trajectory using the instantaneous state as a new intial condition for the solution of the TPBVP. This method results in the "open-loop feedback" control law which for the class of problems in this dissertation is the same as the closed-loop control law.

Cost Functions Special Cases

For the range and angle-off cost functions of Eq (5.35) we have the following limiting cases

(a) $a = 1$ Pursuer considers only final range

(b) $a = 0$ Pursuer considers only angle-off

(c) $b = 1$ Evader considers only final range

(d) $b = 0$ Evader considers only final angle-off

Interesting cases result from (a) and (d) and (b) and (c), and serve to illustrate applications of the NZSDG theory to pursuit-evasion games.

Case I. ($a = 1$, $b = 0$)

This case occurs when the pursuer's goal is to minimize the relative

range while the evader's goal is out-turning the pursuer. The cost functions of Eq (5.35) become

$$J^p = R\ (t_f)$$
$$J^e = -\theta^2\ (t_f) \tag{5.43}$$

The θ angles of interest are $-\pi \leq \theta \leq \pi$. To solve a given initial value problem using the cost functions, state equations and influence function equations in Eqs (5.43), (5.8) and (5.20), a TPBVP must be solved. The alternative is to examine the backward solutions from the terminal surface for clues to the solutions' behavior. The latter approach will be followed for this problem.

Transversality conditions yield the terminal values for the influence functions

$$\lambda^p_{xp}\ (t_f) = [-(x^e - x^p)/R]_{t=t_f}$$

$$\lambda^p_{yp}\ (t_f) = [-(y^e - y^p)/R]_{t=t_f}$$

$$\lambda^p_{Vp}\ (t_f) = 0$$

$$\lambda^p_{\gamma p}\ (t_f) = 0$$

$$\lambda^p_{xe}\ (t_f) = -\lambda^p_{xp}\ (t_f)$$

$$\lambda^p_{ye}\ (t_f) = -\lambda^p_{yp}\ (t_f)$$

$$\lambda^p_{Ve}\ (t_f) = 0$$

$$\lambda^p_{\gamma e}\ (t_f) = 0$$

$$\lambda^e_{xp}\ (t_f) = [-2\,\theta\,(y^e - y^p)/R^2]_{t=t_f}$$

$$\lambda^{e}_{yp}\ (t_f) = [\ 2\ \theta\ (x^{e}-x^{p})/R^{2}]_{t=t_f}$$

$$\lambda^{e}_{Vp}\ (t_f) = 0$$

$$\lambda^{e}_{\gamma p}\ (t_f) = (2\ \theta)_{t=t_f} \tag{5.44}$$

$$\lambda^{e}_{xe}\ (t_f) = -\ \lambda^{e}_{xp}\ (t_f)$$

$$\lambda^{e}_{ye}\ (t_f) = -\ \lambda^{e}_{yp}\ (t_f)$$

$$\lambda^{e}_{Ve}\ (t_f) = 0$$

$$\lambda^{e}_{\gamma e}\ (t_f) = 0$$

The open-loop equilibrium controls for this problem are given by Eqs (5.21) and (5.22)

$$n^{p*} = -\ n^{p}_{max}\ \text{sgn}\ \lambda^{p}_{\gamma p} \tag{5.21}$$

$$n^{e*} = -\ n^{e}_{max}\ \text{sgn}\ \lambda^{e}_{\gamma e} \tag{5.22}$$

except when the arguments $\lambda^{p}_{\gamma p}$ and/or $\lambda^{e}_{\gamma e}$ are identically zero for a nonzero time interval.

By examining these control equations, the control laws in a region near the terminal surface can be characterized. Examining the pursuer's control first, from Eqs (5.21) and (5.44) we find that at t_f, n^{p*} is undefined since $\lambda^{p}_{\gamma p}\ (t_f) = 0$. We thus require the derivative $\dot{\lambda}^{p}_{\gamma p}\ (t_f)$ to determine $n^{p*}\ (t_f^-)$, where t_f^- is the time an instant before reaching the terminal surface,

$$n^{p*}\ (t_f^-) = n^{p}_{max}\ \text{sgn}\ [\dot{\lambda}^{p}_{\gamma p}\ (t_f)\ ] \tag{5.45}$$

$$= n^{P}_{max} \ \text{sgn} \ [\lambda^{P}_{xp} V^{P} \sin \gamma^{P} - \lambda^{P}_{yp} V^{P} \cos \gamma^{P}]_{t=t_f}$$

$$= n^{P}_{max} \ \text{sgn} \ [\ \sin \theta \ (t_f)]$$

Eq (5.45) implies that the pursuer's control in a neighborhood of the terminal surface is determined by the line of sight angle θ. At t_f the pursuer employs his control so as to rotate his velocity vector toward the line of sight. We now show that this situation holds for a larger region and not just at t_f.

In Eq (5.45) $\dot{\lambda}^{P}_{\gamma p}$ can be expressed as

$$\dot{\lambda}^{P}_{\gamma p} = V^{P} \sin \ (\gamma^{P}_{f} + \theta_f - \gamma^{P}) \tag{5.46}$$

where

$$\gamma^{P}_{f} \equiv \gamma^{P} \ (t_f) \qquad \theta_f \equiv \theta \ (t_f)$$

Since $\lambda^{P}_{\gamma p} \ (t_f) = 0$, $\lambda^{P}_{\gamma p}$ is opposite in sign to $\dot{\lambda}^{P}_{\gamma p} \ (t_f)$ for a period of time before reaching the terminal surface. If V^{P} remains nearly constant $\dot{\lambda}^{P}_{\gamma p}$ is nearly periodic, hence this period of time $[t_f - t_1]$ is determined approximately by setting

$$\int_{t_1}^{t_f} \dot{\lambda}^{P}_{\gamma p} \, dt = 0 \tag{5.47}$$

and solving for t_1. Fig. 13 illustrates typical behavior of $\lambda^{P}_{\gamma p}$ vs. γ^{P} in the interval $[t_1, t_f]$.

A similar analysis for the evaders control n^{e*} results in equations analogous to Eqs (5.45) and (5.46),

$$n^{e*} \ (t_f^-) = n^{e}_{max} \ \text{sgn} \ [\frac{2\theta_f V^{e}_{f}}{R_f} \cos \ (\gamma^{P}_{f} + \theta_f - \gamma^{e}_{f})] \tag{5.48}$$

Eq (5.48) implies that the evader's control in a neighborhood of the

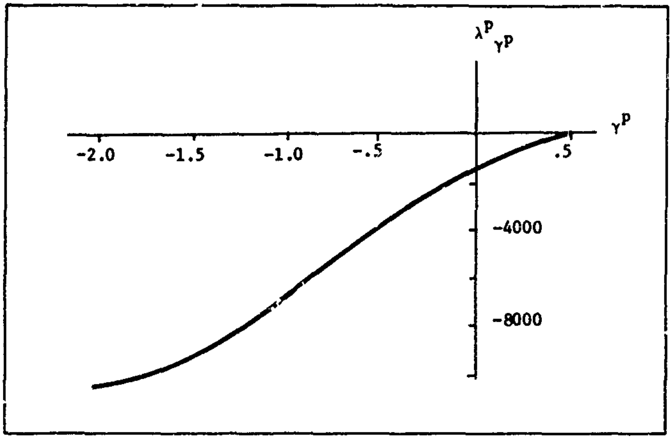

Fig. 13. Typical Behavior of $\lambda^p_{\gamma^p}$ vs. γ^p

terminal surface is determined by the line of sight angle θ and the flight path angles of both players γ^p and γ^e. The evader employs his control law so as to rotate the line of sight away from the velocity vector of the pursuer. We now show that this situation also holds in a larger region.

$\dot{\lambda}^e_{\gamma^e}$ can be expressed as

$$\dot{\lambda}^e_{\gamma^e} = \frac{2\,\theta_f}{R_f} V^e \cos(\gamma^p_f + \theta_f - \gamma^e) \tag{5.49}$$

Since $\lambda^e_{\gamma^e}(t_f) = 0$, $\lambda^e_{\gamma^e}$ is opposite in sign from $\dot{\lambda}^e_{\gamma^e}(t_f)$ for a period of time before reaching the terminal surface. As in Eq (5.47) this period of time $[t_2, t_f]$ is determined approximately, provided V^e is nearly constant, by integrating

$$\int_{t_2}^{t_f} \cos(\gamma^p_f + \theta_f - \gamma^e)\, dt = 0 \tag{5.50}$$

and solving for t_2.

Fig. 14 illustrates typical behavior of $\lambda^e_{\gamma e}$ vs. γ^e backwards from the terminal surface. As long as $\lambda^e_{\gamma e}$ has the same sign, the control n^{e*} is constant. The trajectories for p and e associated with Figs. 13 and 14 are shown in Fig. 15. For this problem the singular solutions appear to be pathological since the evader's cost function is defined so as to avoid the tail chase situation or the pursuer singular arc where θ (t_f) is zero. Therefore, we may conclude that the important control laws in this special case are those closed-loop laws where the pursuer employs his control to force the angle-off angle θ to zero and the evader employs his control to force the angle θ away from zero (hopefully to π radians.)

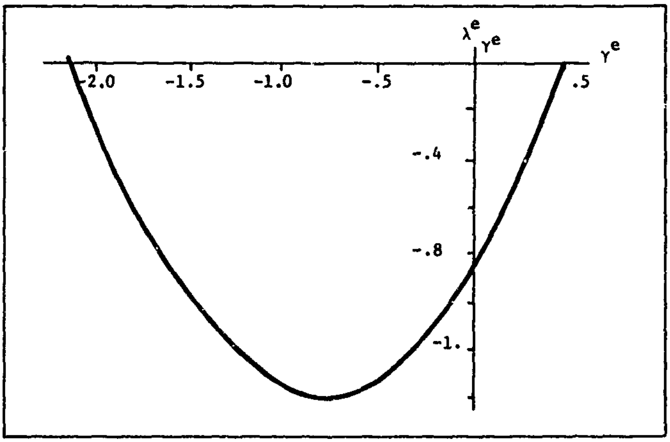

Fig. 14. Typical Behavior of $\lambda^e_{\gamma e}$ vs. γ^e

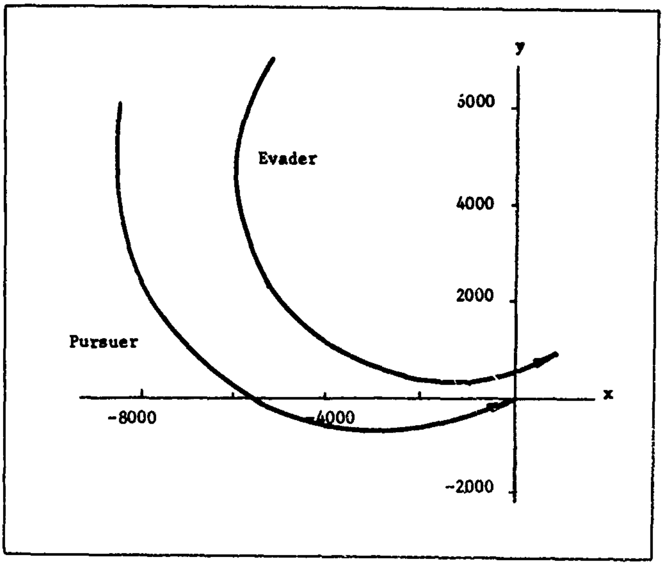

Fig. 15. Typical Trajectories for the Special Case
$a = 1, b = 0$

Case II. $(a = 0, b = 1)$

In this case the pursuer considers only the final angle-off angle $\theta\ (t_f)$ and the evader considers only the final range $R\ (t_f)$. The cost functions Eq (5.35) become

$$J^p = \theta^2\ (t_f) \qquad (5\ 51)$$

$$J^e = -R\ (t_f)$$

Again we will examine the backward trajectories to deduce the behavior of the solutions.

Transversality conditions yield the terminal values for tha influence functions

$$\lambda^p{}_{x^p}\ (t_f) = [\ 2\ \theta\ (y^e - y^p)/R^2]_{t=t_f}$$

$$\lambda^{p}_{y^p}\ (t_f) = [\ -2\ \theta(x^e - x^p)/R^2]_{t=t_f}$$

$$\lambda^{p}_{V^p}(t_f) = 0$$

$$\lambda^{p}_{\gamma^p}\ (t_f) = -2\ \theta\ (t_f)$$

$$\lambda^{p}_{x^e}\ (t_f) = -\ \lambda^{p}_{x^p}\ (t_f)$$

$$\lambda^{p}_{y^e}\ (t_f) = -\ \lambda^{p}_{y^p}\ (t_f)$$

$$\lambda^{p}_{V^e}\ (t_f) = 0$$

$$\lambda^{p}_{\gamma^e}\ (t_f) = 0 \qquad (5.52)$$

$$\lambda^{e}_{x^p}\ (t_f) = [(x^e - x^p)/R]_{t=t_f}$$

$$\lambda^{e}_{y^p}\ (t_f) = [(y^e - y^p)/R]_{t=t_f}$$

$$\lambda^{e}_{V^p}\ (t_f) = 0$$

$$\lambda^{e}_{\gamma^p}\ (t_f) = 0$$

$$\lambda^{e}_{x^e}\ (t_f) = -\ \lambda^{e}_{x^p}\ (t_f)$$

$$\lambda^{e}_{y^e}\ (t_f) = \ \ \lambda^{e}_{y^p}\ (t_f)$$

$$\lambda^{e}_{V^e}\ (t_f) = 0$$

$$\lambda^{e}_{\gamma^e}\ (t_f) = 0$$

The open loop equilibrium controls for this problem are the same as Eqs (5.21) and (5.22), except for the possibility of singular controls. From the requirements for a pursuer terminal singular trajectory

$$n^{p*} = 0 \qquad (5.53)$$

and

$$\lambda^p_{\gamma^p} = 0 \tag{5.54}$$

which in turn implies from Eq (5.52) that

$$\theta\ (t_f) = 0 \tag{5.55}$$

Eq (5.55) implies that the pursuers' velocity vector is on the line of sight to the evader at t_f. The evaders' singular arc is characterized by

$$n^{e*} = 0 \tag{5.56}$$

and

$$\lambda^e_{\gamma^e} = 0 \tag{5.57}$$

We have already shown that the evader's velocity vector must also lie on the line of sight between pursuer and evader at t_f in order for e to be on a terminal singular arc. We therefore conclude that the singular trajectories for this special case have the same general character as those in Figs. 10, 11, and 12.

Two special cases of the cost functions in Eq (5.35) have been examined. In Case I where the pursuer considers final range only and the evader considers final angle-off only it is found that the singular surfaces play an insignificant role in the problem solution due to the fact that the evader's goal avoids the situation where the final angle-off becomes zero. On the other hand, in Case II where the pursuer considers only final angle-off and the evader considers only final range, the singular surfaces appear and do have a significant role in the solutions.

The problem of this chapter illustrates that NZSDG theory can be used to model a pursuit-evasion combat situation in which the goals

of the players are not posed as being diametrically opposite as in zero sum games. The solutions are optimal for each player in the sense of equilibrium optimality.

VI. Conclusions

In this dissertation the theory of Nonzero Sum Differential Games is extended so that the theory can be applied to some combat problems between two or more combatants. Useful transversality conditions are derived in Appendix C to augment the existing theory presented in Chapter II.

A major difficulty in solving NZSDG problems is that in the equilibrium solution the partial derivatives of the value functions and the influence functions can be discontinuous on switching surfaces where the controls are discontinuous. However, for a certain class of NZSDG in which the state equation is nonlinear with the controls bounded and appearing linearly, the partial derivatives of the value functions, and the influence functions are continuous across the switching surfaces. In Chapter III this class of problems is presented and a theorem to test the continuity of the influence functions on singular surfaces is derived. This is a significant result since the continuity of the influence functions makes the equilibrium solution much easier to obtain.

Chapter IV presents two intercept problems in a plane between first two then three combatants. The problems are posed as NZSDG's in which the players are first one then two defenders whose goal is to intercept the attacking player in minimum time. The attacking player's goal is minimization of his range to a fixed position target before intercept occurs. The problem is shown to satisfy the influence function continuity theorem of Chapter III. The solutions are characterized by each player making a hard turn to a particular heading then making a straight dash on that heading until intercept occurs. When the initial turns are neglected, totally singular solutions result which can be easily obtained

numerically. An algorithm is presented which solves the totally singular problems.

Chapter V presents a pursuit-evasion problem of two aircraft in which the goals of the players are not diametrically opposed. The cost functions of each player are different (but not opposite) functions of the final range and angle-off. This problem is also shown to satisfy the influence function continuity theorem of Chapter III. The solutions to this problem are characterized by trajectories which consist of hard turns and straight dashes. Two special cases are discussed in which first the pursuer considers only final range and the evader considers only final angle-off. In this case the singular arc (straight dash) appears to have no significant role in solutions due to the nature of the evader's cost function. In the second special case the pursuer considers only final angle-off while the evader considers only final range. The usual hard turns and straight dashes for both players appear in solutions for this case.

The results of this dissertation are significant because the use of a NZSDG formulation to model combat situations results in a more general problem in which the goals of the players are not required to be diametrically opposite as in zero sum differential games. This means that conflict situations can be modeled with more flexibility and realism to reflect the actual goals of the opponents.

Many problems involving the complex aircraft in today's Air Force inventory can be adequately modeled using the class of problems presented in Chapter III and investigated in this dissertation. The state equation of Chapter III, although linear in the control variables, often serves as a close approximation to the state equation with nonlinear control

variables, and yields solutions which are close approximations of the nonlinear solution. For this reason,the combat models which fit the class of problems investigated give a basic insight into the nature of solutions involving the more complex models with nonlinear control variables in their state equation.

The author intends to investigate the extent to which the models of aerodynamic vehicles can be linearized in the control variables so that results of this dissertation may be applied. In addition, it is the author's intent to investigate the possibility of extending the class of problems studied in this dissertation to include the fully nonlinear models of aerodynamic vehicles.

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Appendix A

Cost Functions

The selection of meaningful cost functions is one of the most important details in formulating any differential game, for these cost functions influence the control strategies, the trajectory, and the outcome of the game. Quantities to be weighted in a cost function are generally either point functions such as terminal miss distance or path functions which require accummulation by integration along a trajectory. The following sections describe some meaningful cost functions for a NZSDG.

Terminal Miss Distance

If player i's goal is to minimize the difference between his state (or components thereof) and player j's state at the termination of the game, his choice of cost functions can be

$$J^i = || x^i(t_f) - x^j(t_f) ||_{Q^i} \qquad (A.1)$$

where x^i and x^j are the state components of the i^{th} and j^{th} players respectively; Q^i is a positive semi-definite weighting matrix used to weight the state components of interest.

Minimum Time

If the goal of plaver i is to minimize the time to game termination his choice of cost functions is either

$$J^i = t_f \qquad (A.2)$$

or

$$J^i = \int_{t_o}^{t_f} dt \qquad (A.3)$$

where t_f is the smallest time such that the termination criteria is satisfied.

Minimum Control Effort (minimum fuel)

Player i's goal of game termination with the least control effort is reflected in the integral cost function

$$J^i = \int_{t_o}^{t_f} || U^i ||_{P^i_F(t)} \, dt \tag{A.4}$$

where $P^i{}_F$ is a positive semi-definite weighting matrix function.

Minimum Energy

Player i's goal of game termination with the least expenditure of energy is reflected in the cost function

$$J^i = \int_{t_o}^{t_f} || U^i ||^2_{P^i_E(t)} \, dt \tag{A.5}$$

where $P^i{}_E$ is a positive semi-definite weighting matrix function. It should be emphasized that caution and judgment must be used whenever a combination of various goals are included in the same cost function. The goal of minimizing time as in Eq (A.2) and the goal of minimizing fuel or energy as in Eqs. (A.4) and (A.5) are directly conflicting, for generally, minimizing time requires maximum effort. An example of this situation is the cost function

$$J^i = a \, t_f + (1-a) \int_{t_o}^{t_f} || U^i ||_{P^i_F(t)} \, dt \tag{A.6}$$

where $0< a < 1$.

Since both goals cannot be met simultaneously, the choice of the weighting parameter a becomes a matter of judgment based on knowledge of the problem.

Appendix B

Formal Derivation of the Influence Function Equations From the HJB Equations.

This appendix presents the author's formal derivation of influence function equations Eq (2.20) from the generalized HJB equations Eq (2.18). The derivation is a generalization of Dreyfus's work in Ref (10). Assume a normal problem. The HJB equations written in the form of Eq (2.14) are

$$W^i_t = - \min_{U^i} \left[W^i_x \, f\,(x,\, t,\, U) + L^i\,(x,\, t,\, U) \right] \quad i = 1, \ldots, N \tag{B.1}$$

where

$$U = (U^1, \ldots, U^N) \tag{B.2}$$

Assume U^i is bounded according to the equation

$$K\,(U^i) \leq 0 \tag{B.3}$$

If the partial derivatives W^i_t and W^i_x are known for a particular point (x, t), then theoretically the equilibrium controls U* can be determined from Eq (B.1) in terms of x and t. If the equations for $\dot{W}^i_x$ and $\dot{W}^i_t$ are known, then equilibrium trajectories can be constructed forward in time from the point (x, t), the controls being determined from the known values of W^i_x, W^i_t, x and t in Eq (B.1). Proceeding formally, the equation for $\dot{W}^i_x$ will be obtained. $\dot{W}^i_x$ can be expressed as

$$\dot{W}^i_x = W^i_{xx}\dot{x} + W^i_{xt} \qquad i = 1, \ldots, N \tag{B.4}$$

$$= W^i_{xx}\, f + W^i_{xt}$$

Assume that the equilibrium control vector U^* is known in the form

$$U^* = U^* (x, t) \tag{B.5}$$

Then Eq (B.1) can be written

$$W^i{}_t = -W^i{}_x f (x, t, U^*) - L^i (x, t, U^*) \quad i = 1, \dots, N \tag{B.6}$$

Taking a partial derivative w.r.t. x of both sides in Eq (B.6) yields

$$W^i{}_{tx} = -W^i{}_{xx} f - W^i{}_x (f_x + f_{U^*} U^*{}_x) - (L^i{}_x + L^i{}_{U^*} U^*{}_x) \tag{B.7}$$

Assuming $W^i{}_{xt} = W^i{}_{tx}$ and substituting for $W^i{}_{xt}$ into Eq (B.4) from Eq (B.7) results in the equation

$$\dot{W}^i{}_x = - (W^i{}_x f_x + L^i{}_x) - (W^i{}_x f_{U^*} + L^i{}_{U^*}) U^*{}_x \quad i = 1, \dots, N \tag{B.8}$$

or

$$\dot{W}^i{}_x = - (W^i{}_x f_x + L^i{}_x) - \sum_{j=1}^{N} (W^i{}_x f_{U^{j*}} + L^i{}_{U^{j*}}) U^{j*}_x \tag{B.9}$$

$$i = 1, \dots, N$$

Suppose that U^i is interior to its constraint region defined in Eq (B.3); then the minimizing U^i in Eq (B.1) satisfies the equation

$$W^i{}_x f_{U^{i*}} + L^i{}_{U^{i*}} = 0 \tag{B.10}$$

Substituting from Eq (B.10) into Eq (B.9) gives the equation for $\dot{W}^i{}_x$

$$\dot{W}^i{}_x = - (W^i{}_x f_x + L^i{}_x) - \sum_{\substack{j=1 \\ j \neq i}}^{N} (W^i{}_x f_{U^{j*}} + L^i{}_{U^{j*}}) U^{j*}_x \tag{B.11}$$

$$i = 1, \dots, N$$

Defining

$$H^i = L^i + W^i{}_x\, f \qquad \text{(B.12)}$$

and on an equilibrium trajectory defining

$$\lambda^i \equiv W^i{}_x \qquad \text{(B.13)}$$

Eq (B.11) becomes Eq (2.20) of Chapter II, the influence function equations,

$$\dot{\lambda}^i = - H^i{}_x - \sum_{\substack{j=1 \\ j\neq i}}^{N} H^i{}_{U^{j*}}\, U^{j*}_x \qquad i = 1, \ldots, N \qquad \text{(B.14)}$$

Suppose now that U^i is on the boundary of its constraint region defined in Eq (B.3), then it can be argued as in Ref [10] that $U^i{}_x = 0$. Substituting $U^i{}_x = 0$ into Eq (B.9) and noting that the other controls may not be on their constraint region boundaries so that $U^j{}_x (j \neq i)$ is not necessarily zero, Eqs (B.11) and (B.14) are again obtained. Thus, the influence function differential equation holds whether or not U^{i*} is interior to its constraint region.

The same general procedure as above may be used to show that on an equilibrium trajectory the equation for $\dot{W}^i{}_t$ is

$$\dot{W}^i{}_t = -H^{i*}{}_t - H^{i*}{}_{U*}\, U^*{}_t \qquad \text{(B.15)}$$

If the state equation Eq (2.1) is autonomous then Eq (B.15) reduces to

$$\dot{W}^i{}_t = 0 \qquad i = 1, \ldots, N \qquad \text{(B.16)}$$

Appendix C

Influence Function Transversality Conditions

Three generally used forms of the transversality conditions are presented in this appendix: the Dreyfus form [10], the Isaacs form [17], and the Berkovitz form [5]. The generalization of the first two forms to a NZSDG is due to the author while the third form is due to Sarma [31]. Since all the forms are in common use it was felt that each form should be presented, and the use of any one form is a matter of preference and convenience. Throughout the dissertation, however, only the Dreyfus form is used.

For each form we assume an n-dimensional terminal manifold in the space of x and t. For the Dreyfus and Isaacs form the terminal manifold is described by the scalar equation

$$\Psi\ [\ x\ (t_f),\ t_f] = 0 \tag{2.3}$$

In Eq (2.3) t_f is the smallest time t such that the equation is satisfied. If the final time is fixed, then Eq (2.3) is usually written

$$\Psi = T - t_f = 0$$

where T is a fixed number.

Occasionally Eq (2.3) does not describe the terminal conditions; for example, the terminal condition may be the state reaching a single point (e.g., the origin). In this case a n-dimensional sphere of radius δ with the point at the center may be employed as an n-dimensional terminal surface then a limit taken ($\delta \rightarrow 0$) to determine the transversality conditions [27]. In the case of more than one scalar equation describing the terminal surface it is generally necessary to employ a Lagrange Multiplier technique such as found in reference [10].

Dreyfus Form

This form assumes a NZSDG with terminal cost functions, i.e. $L^i = 0$ in Eq (2.4); so we have

$$J^i = \phi^i[x(t_f), t_f] \tag{C.1}$$

This assumption is not overly restrictive since every problem with a cost function containing an integral can be easily converted to an equivalent terminal cost problem [17]. To construct the equivalent terminal cost problem assume only the i^{th} player has an integral cost function; define another state component x_{n+1} such that $\bar{x} = (x, x_{n+1})$ and

$$\dot{x}_{n+1} = L^i \qquad x_{n+1}(t_o) = 0 \tag{C.2}$$

Then the cost function in the equivalent problem is

$$\bar{J}^i = \phi^i[x (t_f), t_f] + x_{n+1}(t_f) = \bar{\phi}^i[\bar{x} (t_f), t_f] \tag{C.3}$$

We now proceed with the derivation of the Dreyfus form of the transversality conditions.

Suppose the state equation is written

$$\dot{x} = f (x, t, U^*) \qquad x(t_o) = x_o \tag{C.4}$$

Consider a point $x(t) = x_f$, $t = t_f$, such that (x_f, t_f) lie on the terminal manifold; that is

$$\Psi (x_f, t_f) = 0 \tag{C.5}$$

Now consider a variation, δx_f on the terminal manifold. The change in the i^{th} cost is

$$\delta J^i = \phi^i_{x_f} \delta x_f + \dot{\phi}^i(t_f) \Delta t_f \tag{C.6}$$

where Δt_f is the variation in terminal time induced by the variation

δx_f. The notation Δt_f is used to distinguish the independent variation δx_f from the induced variation Δt_f. $\dot{\phi}^i$ is the total time derivative of ϕ^i. Similarly,

$$\delta \Psi = \Psi_{x_f} \; \delta x_f + \dot{\Psi} \, (t_f) \; \Delta t_f \tag{C.7}$$

In order for the equation

$$\Psi \, (\, x_f + \; x_f, \; t_f + \; t_f) = 0 \tag{C.8}$$

to hold to first order,

$$\delta \Psi \; = 0 \tag{C.9}$$

Solving formally in (C.7) for Δt_f we have (assuming $\dot{\Psi}^{-1} \neq 0$)

$$\Delta t_f = - \; \dot{\Psi}^{-1}(t_f) \; \Psi_{x_f} \; \delta x_f \tag{C.10}$$

The influence function λ^i (t) can be regarded as the sensitivity coefficient of the cost J^i to a change in the state x at time t [10], so that at t_f

$$\lambda^i \; (t_f) = \delta J^i \; / \; \delta x_f \tag{C.11}$$

So, from Eqs (C.6) and (C.11) the equation for the influence function variables at the terminal time is

$$\lambda^i \; (t_f) = [\; \phi^i{}_x - (\; \dot{\phi}^i \; / \; \dot{\Psi} \;) \Psi_x \;]_{t=t_f} \qquad i = 1, \; . \; . \; ., \; N \tag{C.12}$$

This equation may also be written in the form

$$\lambda^i \; (t_f) = [\; \phi^i{}_x - (\; \phi^i{}_x \; f + \phi^i{}_t \;) \; / \; (\; \Psi_x \; f + \Psi_t \;) \; \Psi_x \;]_{t=t_f} \tag{C.13}$$

Since f [x, t, U^* (x, $\{\lambda^j\}$, t)] depends on the set $\{\lambda^j\}$ generally in a nonlinear way, the equations in Eqs (C.12) and (C.13) are a set of N coupled nonlinear vector equations in the N unknown vectors λ^i (t_f).

Isaacs Form

This form of the transversality conditions is a direct extension

of the zero sum differential game transversality conditions in reference [17]. There are no restrictions on the form of the cost functions in this method so that cost functions of the form in Eq (2.4) are assumed,

$$J^i = \phi^i\,[x(t_f),\ t_f] + \int_{t_o}^{t_f} L^i\,[x(t),\ t,\ U]\ dt \qquad i = 1, \ldots, N \quad (2.4)$$

The terminal manifold is described by Eq (2.3). Since many readers are familiar with Isaacs' notation we will borrow notation from Ref. [17].

Let the terminal manifold be parameterized by the equations

$$x_i(t_f) = h_i(s_1,\ s_2,\ \ldots,\ s_n) \qquad i = 1, \ldots, n+1 \qquad (C.14)$$

where we have let

$$\begin{aligned} x_{n+1} &= t \\ s &= (s_1,\ s_2,\ \ldots,\ s_n) \\ h &= (h_1,\ h_2,\ \ldots,\ h_{n+1}) \\ \bar{x} &= (x,\ t) \end{aligned} \qquad (C.15)$$

On the terminal surface, the value W^i of the i^{th} player equals the terminal portion of the i^{th} cost function,

$$W^i(t_f) = \phi^i[x(t_f),\ t_f] = \phi^i(s) \qquad (C.16)$$

If vector derivatives of both sides in Eq (C.16) are taken w.r.t. s we have

$$W^i_{\bar{x}}\,\bar{x}_s = \phi^i_s \qquad i = 1, \ldots, N \qquad (C.17)$$

or equivalently, since $\bar{x}_s = h_s$ on the terminal surface

$$\phi^i_s = W^i_{\bar{x}}\,h_s \qquad i = 1, \ldots, N \qquad (C.18)$$

In component notation, Eq (C.18) can be written

$$\phi^i_{s_k} = \sum_{j=1}^{n+1} W^i_{x_j} h_{j_{s_k}} \qquad \begin{array}{l} k = 1, \ldots, n \\ i = 1, \ldots, N \end{array} \qquad (C.19)$$

Eq (C.19) reduces to Isaacs' form [17] if N is set equal to 1, and f and L^i are autonomous (making $W^i_t = W^i_{x_{n+1}} = 0$).

Eq (C.19) provides nN equations in the (n+1)N unknowns $W^i_{x_j}(t_f)$, $i = 1, \ldots, N$; $j = 1, \ldots, n+1$. N additional equations are required to solve the system of equations, and fortunately, they are available. From the HJB equations Eq (2.18) we have

$$W^i_t = -(W^i_x f + L^i) \qquad i = 1, \ldots, N \qquad (2.18)$$

where the equilibrium controls have been substituted into f and L^i. To cast Eq (2.18) into the notation of this Appendix, note that W^i_t in Eq (2.18) is identical to $W^i_{x_{n+1}}$

$$W^i_t \equiv W^i_{x_{n+1}} \qquad (C.20)$$

Let $\bar{f}$ be the augmented vector

$$\bar{f} = \begin{bmatrix} f \\ \hline 1 \end{bmatrix} \qquad (C.21)$$

Then Eq (2.18) can be written

$$W^i_{\bar{x}} \bar{f} + L^i = 0 \qquad i = 1, \ldots, N \qquad (C.22)$$

Evaluating Eq (C.22) on the terminal manifold by substituting the parameters $(s_1, s_2, \ldots, s_n)$ provides the N additional equations necessary to solve for the $W^i_{x_j}(t_f)$. Eqs (C.19) and (C.22) are a set of (n+1)N nonlinear equations in the (n+1)N variables $W^i_{x_j}(t_f)$. Recalling the equivalence of λ^i and W^i_x on an equilibrium trajectory, we see that the solution of Eqs (C.18) and (C.22) gives us the values of $\lambda^i(t_f)$ and

also the values of $W^i_t\ (t_f)$. Caution must be used in the solution since multiple solutions can occur corresponding to termination on different sides of the terminal manifold. In such cases the physical situation dictates which solution is to be used.

Sarma - Berkovitz Form

A form of the transversality conditions which closely resembles Isaacs' has been obtained by Sarma [31] by generalizing the zero sum differential game results of Berkovitz [6]. A few notational changes are required to present the results. In addition, Sarma has a more general form of terminal surface consisting of the union of a finite number of n-dimensional class C^1 manifolds. The terminal surface T is defined by the equation

$$T \equiv \bigcup_{j=1}^{\alpha} T_j \qquad (C.23)$$

where each T_j is an n-dimensional C^1 surface. Each T_j is parameterized by the equations

$$t_f = T_j\ (\sigma)$$
$$x_f = \chi_j\ (\sigma) \qquad (C.24)$$

where the parameter σ is

$$\sigma \equiv (\sigma_1, \sigma_2, \ldots, \sigma_n) \qquad (C.25)$$

The transversality equations are

$$\partial\phi^i/\partial\sigma + H^i\ \ \partial T_j/\partial\sigma - \lambda^i\ \partial\chi_j/\partial\sigma = 0 \qquad (C.26)$$

where the index j refers to the specific T_j on which Eq (C.26) is evaluated. Eq (C.26) together with the HJB equations,

$$W^i_t = -(W^i_x f + L^i) \qquad i = 1, \ldots, N \tag{2.18}$$

evaluated on the terminal surface are sufficient to solve for the unknown λ^i and W^i_t on the terminal surface. (Again recall the equivalence of λ^i and W^i_x on an equilibrium trajectory)

Appendix D

Control Laws on Universal Surfaces

In this appendix the control laws on singular surfaces are derived for the problem in Chapter V. Similar results hold for the problem of Chapter IV. The derivation is based on the necessary condition for singular controls found in Eqs (2.28) - (2.30) repeated here for reference,

$$H^i_{U^i_j} = 0 \qquad \left| H^i_{U^i\,U^i} \right| = 0 \tag{2.28}$$

$$\dot{H}^i_{U^i_j} = \ddot{H}^i_{U^i_j} = \cdots = 0 \tag{2.29}$$

$$(-1)^k\, \partial/\partial U^i_j \,[\, d^{2k}/dt^{2k}\, H^i_{U^i_j}\,] \geq 0 \tag{2.30}$$

Applying these equations, we consider the possibility of a singular pursuer control (a similar argument holds for a singular evader control).

The first necessary condition is from Eq (2.28),

$$\lambda^P_{\gamma P} = 0 \tag{D.1}$$

Similarly, all time derivatives of $\lambda^P_{\gamma P}$ are zero, so that

$$\dot{\lambda}^P_{\gamma P} = \ddot{\lambda}^P_{\gamma P} = \ldots = 0 \tag{D.2}$$

From the influence function equations Eq (5.20), the equation for $\dot{\lambda}^P_{\gamma P}$ is

$$\dot{\lambda}^P_{\gamma P} = \lambda^P_{xP}\, V^P \sin \gamma^P - \lambda^P_{yP}\, V^P \cos \gamma^P \tag{D.3}$$

so that from Eqs (D.2) and (D.3)

$$\lambda^P_{xP} \sin \gamma^P - \lambda^P_{yP} \cos \gamma^P = 0 \tag{D.4}$$

or

$$\tan \gamma^P = \lambda^P_{yP} \,/\, \lambda^P_{xP}$$

Taking a time derivative of both sides of Eq (D.3) and substituting for $\dot{\lambda}^P_{xP}$, $\dot{\lambda}^P_{yP}$, $\dot{V}^P$ and $\dot{\gamma}^P$ from Eqs (5.8) and (5.20) we obtain the equation for $\ddot{\lambda}^P_{\gamma P}$

$$\ddot{\lambda}^P_{\gamma P} = \frac{n^P g}{V^P} \left(V^P \lambda^P_{xP} \cos \gamma^P + V^P \lambda^P_{yP} \sin \gamma^P\right) = 0 \quad \text{(D.5)}$$

From Eq (D.5) there are three possibilities

(a) $\lambda^P_{xP} \cos \gamma^P + \lambda^P_{yP} \sin \gamma^P = 0$

or

(b) $n^P = 0$

or

(c) both (a) and (b)

It is easy to establish that (a) is not possible for from (a)

$$\tan \gamma^P = - \lambda^P_{yP}/\lambda^P_{xP}$$

which is a contradiction to Eq (D.4). (b) is the only remaining possibility so that on a singular surface for the pursuer the equilibrium control candidate n^{P*} is

$$n^{P*} = 0 \quad \text{(D.6)}$$

A similar result holds for the evader's singular control if it occurs,

$$n^{e*} = 0 \quad \text{(D.7)}$$

Application of the necessary condition Eq (2.30) for a singular control for the pursuer results in the requirement

$$\lambda^P_{xP} \cos \gamma^P + \lambda^P_{yP} \sin \gamma^P \leq 0 \quad \text{(D.8)}$$

Eq (D.8) must be satisfied on the pursuer's singular surface. Similarly

for an evader singular control, the inequality

$$\lambda^{e}_{x^e} \cos \gamma^{e} + \lambda^{e}_{y^e} \sin \gamma^{e} \leq 0 \tag{D.9}$$

must hold on the evader's singular surface. It is easily verified that Eqs (D.8) and (D.9) are satisfied with strict inequality on terminal singular trajectories for the pursuer and evader respectively.

Appendix E

Transversality Conditions for the Three Player Problem

In this appendix the vector equation for the transversality conditions in the three player problems of Chapter V is expanded into component form. From Eq (2.25) the transversality condition is

$$\lambda^{s}\,(t_f) = [\,\phi^{s}_{x} - (\dot{\phi}^{s}/\dot{\psi})\,\psi_{x}]_{t=t_f} \qquad s = a,\ c,\ d \tag{E.1}$$

where the cost functions for players a, c, and d are (see Eqs (2.4) and (4.4))

$$J^{c} = J^{d} = t_f \tag{E.2}$$

$$J^{a} = 1/2\ [(x^{a}-x_T)^2 + (y^{a}-y_T)^2] \tag{E.3}$$

The termination criteria from Eq (4.29) is

$$\begin{aligned}\psi[x(t_f),\ t_f] &= 1/2\ [\psi^{d}]\ [\psi^{c}] \\ &= 1/2\ [(x^{d}-x^{a})^2 + (y^{d}-y^{a})^2 - \ell^2] \\ &\qquad [(x^{c}-x^{a})^2 + (y^{c}-y^{a})^2 - k^2] \\ &= 0\end{aligned} \tag{E.4}$$

The state vector x in component form is

$$x^{T} = (x^{d},\ y^{d},\ \gamma^{d},\ x^{c},\ y^{c},\ \gamma^{c},\ x^{a},\ y^{a},\ \gamma^{a}) \tag{E.5}$$

The partial and total derivatives in Eq (B.1) are

$$\phi^{c}_{x} = \phi^{d}_{x} = 0 \tag{E.6}$$

$$\phi^{a}_{x} = (0,\ 0,\ 0,\ 0,\ 0,\ 0,\ x^{a}-x_T,\ y^{a}-y_T,\ 0) \tag{E.7}$$

$$\psi^T_x = \begin{bmatrix} (x^d-x^a)\ \psi^c \\ (y^d-y^a)\ \psi^c \\ 0 \\ (x^c-x^a)\ \psi^d \\ (y^c-y^a)\ \psi^d \\ 0 \\ -(x^d-x^a)\ \psi^c-(x^c-x^a)\ \psi^d \\ -(y^d-y^a)\ \psi^c-(y^c-y^a)\ \psi^d \\ 0 \end{bmatrix} \tag{E.8}$$

$$\dot{\phi}^c = \dot{\phi}^d = 1 \tag{E.9}$$

$$\dot{\phi}^a = \phi^a_x\ f$$

$$= (x^a-x_T)\ V^a \cos\gamma^a + (y^a-y_T)\ V^a \sin\gamma^a \tag{E.10}$$

where f is defined from the state equation,

$$f^T = (V^d \cos\gamma^d,\ V^d \sin\gamma^d,\ c^d u,\ V^c \cos\gamma^c,\ V^c \sin\gamma^c,\ c^c w, \tag{E.11}$$

$$V^a \cos\gamma^a,\ V^a \sin\gamma^a,\ c^a v)$$

Finally,

$$\dot{\psi} = 1/2\ [\dot{\psi}^d\ \psi^c + \psi^d\ \dot{\psi}^c] = \psi_x\ \ f \tag{E.12}$$

$$= (x^d-x^a)\ (V^d \cos\gamma^d - V^a \cos\gamma^a)\ \psi^c +$$

$$(y^d-y^a)\ (V^d \sin\gamma^d - V^a \sin\gamma^a)\ \psi^c +$$

$$(x^c-x^a)\ (V^c \cos\gamma^c - V^a \cos\gamma^a)\ \psi^d +$$

$$(y^c-y^a)\ (V^c \sin\gamma^c - V^a \sin\gamma^a)\ \psi^d$$

The equations for $\dot{\psi}^c$ and $\dot{\psi}^d$ are

$$\dot{\psi}^c = 2\ [(x^c-x^a)\ (V^c \cos \gamma^c - V^a \cos \gamma^a) + (y^c-y^a)\ (V^c \sin \gamma^c - V^a \sin \gamma^a)] \tag{E.13}$$

$$\dot{\psi}^d = 2\ [(x^d-x^a)\ (V^d \cos \gamma^d - V^a \cos \gamma^a) + (y^d-y^a)\ (V^d \sin \gamma^d - V^a \sin \gamma^a)] \tag{E.14}$$

The equation for $\ddot{\psi}$ which is required in applying l'Hospital's rule in the case of simultaneous intercept is

$$\ddot{\psi} = 1/2\ [\ \ddot{\psi}^d\ \psi^c + \dot{\psi}^d\ \dot{\psi}^c + \dot{\psi}^d\ \dot{\psi}^c + \psi^d\ \ddot{\psi}^c] \tag{E.15}$$

In the case of simultaneous intercept,

$$\psi^c = \psi^d = 0 \tag{E.16}$$

and $\ddot{\psi}$ in Eq (B.16) becomes

$$\ddot{\psi} = \dot{\psi}^d\ \dot{\psi}^c \tag{E.17}$$

Vita

Anthony Lynn Leatham was born on 30 May 1937 in Wellsville, Utah. He graduated from High School in Burley, Idaho in 1955, attended Utah State University for one year, then entered the Regular Army wherein he obtained an appointment to the U.S. Military Academy, graduating with a Bachelor of Science degree in 1962. He attended the University of Southern California graduating in 1964 with a Master of Science in both Aerospace Engineering and Mechanical Engineering. He served as a project engineer with Headquarters, Air Force Satellite Control Facility, Los Angeles Air Force Station, Los Angeles, California until entering the Air Force Institute of Technology in July 1967. Upon being admitted to candidacy for the Ph.d degree in September 1969 he was assigned as an Aerospace Engineer in the High Speed Aero Performance Branch, Flight Mechanics Division, Air Force Flight Dynamics Laboratory, Wright-Patterson AFB, Ohio.

This dissertation was typed by Miss Marcia Tanner